

Conditional Volatility Estimation by Conditional Quantile Autoregression

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Abstract

This paper considers the problem of estimating conditional volatility function using conditional quantile autoregression function. We estimate the interquantile autoregression range and the conditional volatility function under known distributional assumptions. The conditional volatility function estimator is found to be theoretically consistent. A small simulation study ascertains that the Volatility Estimator is consistent.

Mathematics Subject Classification: 62G05; 62M1

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1. INTRODUCTION

Let $Y_t \in \mathbb{R}$ be F_t -measurable and $X_t \in \mathbb{R}^n$ be F_{t-1} -measurable derived from a stationary and α -mixing multivariate time series $\{U_t, t \in Z\}$ adapted to the sequence

$F_t, -\infty < t < \infty$, of sigma algebras. The response variable $Y_t \in \mathbb{R}$ and the covariate $X_t \in \mathbb{R}^n$ are partitions of U_t such that $U_t = (Y_t, X_t)$. A time series with these feature is an autoregressive series where the regressors are past variables of the series Y_t , that is, Y_t is regressed on $X_t = (Y_{t-1}, Y_{t-2}, \dots)$. For example, the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) process by Bollerslev (1986). We assume that there exists a conditional distribution $G_{X_t}(Y_t)$, from which $\{Y_t, X_t\}$ is drawn from at the point (y, x) . Having in mind of an autoregressive process we consider a process with an AR process for the mean and GARCH process for the scale. To incorporate robustness in the model, we include quantile regression notion in our work which was introduced by Koenker and Basset (1978). For $\theta \in (0,1)$ we define the Quantile Autoregression – Generalized Autoregression Conditional Heteroscedasticity (QAR-GARCH) process according Mwita (2003), which is algebraically expressed as;

$$Y_t = \pi_\theta(X_t) + \sigma(X_t)Z_t \quad (1)$$

Where; $X_t = (Y_{t-1}, Y_{t-1}, \dots, Y_{t-n})$, Z_t are assume *iid* with Zero θ -quantile and unit scale, $\pi_\theta(X_t)$ is the conditional θ -quantile of Y_t given X_t and $\sigma(X_t)$ is the conditional quantile scale function of Y_t given X_t . Since $G(y|x) = G_x(y) = P(Y_t \leq y | X_t = x)$, the conditional quantile function of Y_t given X_t is defined as; $\pi_\theta(x) = \pi(x, \beta_\theta) = \inf\{y \in \mathbb{R} | G(y|x) \geq \theta\}$. The main course in this paper is to estimate the volatility function $\sigma(X_t)$. The Quantile Autoregression-Generalized Autoregression Conditional Heteroscedasticity (QAR-GARCH) process can be rewritten as; $Y_t = \pi(X_t, \beta_\theta) + \sigma(X_t, \alpha)Z_t$. We shall consider a QAR(r) – GARCH(p, q) process defined as;

$$\left. \begin{aligned} Y_t &= \beta_{0,\theta} + \beta_{1,\theta}Y_{t-1} \dots + \beta_{p,\theta}Y_{t-r} + \left(\sqrt{\omega + \alpha_{1,\theta}Y_{t-1}^2 + \dots + \alpha_{p,\theta}Y_{t-p}^2 + \gamma_{1,\theta}\sigma_{t-1}^2 + \dots + \gamma_{q,\theta}\sigma_{t-q}^2} \right) Z_t \\ &= \beta'_\theta X_t + \sqrt{\omega + \alpha'_\theta X_t^2 + \gamma'_\theta \sigma_t^2} Z_t \end{aligned} \right\} \quad (2)$$

Where $\beta_\theta = (\beta_{0,\theta}, \beta_{1,\theta}, \dots, \beta_{r,\theta})'$, $\alpha_\theta = (\alpha_{1,\theta}, \dots, \alpha_{p,\theta})'$, $\gamma_\theta = (\gamma_{1,\theta}, \dots, \gamma_{q,\theta})'$, $X_t = (1, Y_{t-1}, \dots, Y_{t-r})'$, $X_t^2 = (1, Y_{t-1}^2, \dots, Y_{t-p}^2)'$, $\sigma_t^2 = (1, \sigma_{t-1}^2, \dots, \sigma_{t-q}^2)'$ and $Q_\theta(\cdot)$ denotes a quantile of a random variable at θ -level.

Assumption 1

We assume that the conditional quantile for the variable is zero for all quantiles.

That is, for $\theta \in (0,1)$,

$$Q_\theta(Z_t) = 0, \quad \forall \theta$$

At each and every θ -level, the conditional quantile of Y_t given \tilde{X}_t is given by;

$Q_\theta(Y_t|\tilde{X}_t) = \beta'_\theta \tilde{X}_t$. We impose the constraint that $|\sum_{i=1}^q \beta_{i,\theta}| < 1$, so as to achieve stationarity of the QAR (r) as well as for the entire QAR (r)-GARCH (p, q) process. The process in (1) can be expressed as:

$$Y_t = \pi(\tilde{X}_t, \beta_\theta) + \sigma(\tilde{X}_t, \alpha) Z_t \Rightarrow Y_t - \pi(\tilde{X}_t, \beta_\theta) = \sigma(\tilde{X}_t, \alpha) Z_t \tag{3}$$

From (3) $Y_t - \pi(\tilde{X}_t, \beta_\theta)$ denote the residuals. In mind we aim at parameters estimation by minimization of the residuals. Factoring in the notion of leverage, we consider an asymmetric check function so that the weights for negative and positive residuals differ. We define an asymmetric loss function of the form;

$$\vartheta_\theta(Y_t - \pi(\tilde{X}_t, \beta_\theta)) = \begin{cases} \theta(Y_t - \pi(\tilde{X}_t, \beta_\theta)), & \text{if } Y_t - \pi(\tilde{X}_t, \beta_\theta) > 0 \\ (\theta - 1)(Y_t - \pi(\tilde{X}_t, \beta_\theta)), & \text{if } Y_t - \pi(\tilde{X}_t, \beta_\theta) \leq 0 \end{cases} \tag{4}$$

Where; $\vartheta_\theta(x) = x(\theta - I(x < 0))$, for $x \in \mathbb{R}$, with $I(\cdot)$ being the indicator function. Using (3), (4) can be expressed as;

$$\begin{aligned} & \theta |Y_t - \pi(\tilde{X}_t, \beta_\theta)|^+ + (1 - \theta) |Y_t - \pi(\tilde{X}_t, \beta_\theta)|^- \\ & = \sigma(\tilde{X}_t, \alpha) [\theta |Z_t|^+ + (1 - \theta) |Z_t|^-] \end{aligned} \tag{5}$$

Where $|\cdot|^+, |\cdot|^-$ denotes the positive and the negative parts respectively. The quantile autoregression model for residuals (5), can be expressed as an asymmetric loss function as;

$$\vartheta_\theta(Y_t, \pi(\tilde{X}_t, \beta_\theta)) = \sigma(\tilde{X}_t, \alpha) + \sigma(\tilde{X}_t, \alpha) (\vartheta_\theta(Z_t, 0) - 1) \tag{6}$$

With conditional quantile function, $\pi(\tilde{X}_t, \beta_\theta)$ and the noise term, Z_t in this case been given by; $\sigma(\tilde{X}_t, \alpha) \in \mathbb{R}_+$ and $(\vartheta_\theta(Z_t, 0) - 1)$ respectively. Having the loss function defined, we let $\aleph(x)$ denote an objective function which minimizes the expected loss

for our proposed model, defined as: $\aleph(x) = E(\vartheta_\theta(x))$, $x \in \mathbb{R}$ and E is the usual expectation operator.

$$\aleph\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right) = \begin{cases} \theta E\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right), & \text{if } Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right) > 0 \\ (\theta - 1)E\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right), & \text{if } Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right) \leq 0 \end{cases} \quad (7)$$

Definition 1

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz continuous if there exists a constant M such that $\forall x, y \in \mathbb{R}^n$, $\|f(x) - f(y)\| \leq M \|x - y\|$.

To check whether our function is Lipschitz continuous we denote the loss function $\vartheta_\theta\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right) = \vartheta_\theta(Y, \pi)$ (for notation convenience).

Theorem 1

For (Y, π) a real valued random variable, the function $\vartheta_\theta(Y, \pi)$ is Lipschitz continuous in π with Lipschitz constant $M = 1$. i.e. $|\vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2)| \leq |\pi_1 - \pi_2| \forall Y, \pi_1, \pi_2$

Proof

By definition; $\vartheta_\theta(Y, \pi) = (Y - \pi)[\theta - I(Y - \pi \leq 0)]$

Then, let $\pi_1, \pi_2 \in \pi$

$$\begin{aligned} \vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2) &= ((Y - \pi_1)[\theta - I(Y - \pi_1 \leq 0)]) - ((Y - \pi_2)[\theta - I(Y - \pi_2 \leq 0)]) \\ &= \theta(\pi_2 - \pi_1) - [(Y - \pi_1)I(Y - \pi_1 \leq 0) - (Y - \pi_2)I(Y - \pi_2 \leq 0)] \end{aligned}$$

For $\pi_1 < Y < \pi_2$ we have;

$I(Y - \pi_1 \leq 0) = 0$ and $I(Y - \pi_2 \leq 0) = 1$ hence;

$$\begin{aligned} \vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2) &= \theta(\pi_2 - \pi_1) + (Y - \pi_2) \\ &= \theta(\pi_2 - \pi_1) + (Y - \pi_2) + (\pi_1 - \pi_1) \\ &= (Y - \pi_1) - (1 - \theta)(\pi_2 - \pi_1) \end{aligned}$$

For $(Y - \pi_1) > 0$ and $(Y - \pi_2) > 0$, we have;

$$-(1 - \theta)(\pi_2 - \pi_1) \leq \vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2) \leq \theta(\pi_2 - \pi_1)$$

Thus $|\vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2)|$ is bounded from above by at least $\theta(\pi_2 - \pi_1)$ and

$$(1 - \theta)(\pi_2 - \pi_1).$$

Similarly if;

(a) $\pi_1 \leq \pi_2 < Y$, then $I(Y - \pi_1 \leq 0) = 0$ and $I(Y - \pi_2 \leq 0) = 0$ hence;

$$\vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2) = \theta(\pi_2 - \pi_1)$$

(b) $Y < \pi_1 \leq \pi_2$, then $I(Y - \pi_1 \leq 0) = 1$ and $I(Y - \pi_2 \leq 0) = 1$ hence;

$$\vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2) = (\theta - 1)(\pi_2 - \pi_1) = (1 - \theta)|\pi_1 - \pi_2|$$

$$|\vartheta_\theta(Y, \pi_1) - \vartheta_\theta(Y, \pi_2)| \leq \max(\theta, \theta - 1)|\pi_2 - \pi_1|$$

$$\leq |\pi_2 - \pi_1| = |\pi_1 - \pi_2|$$

$$\leq |\pi_1 - \pi_2|$$

Hence its *Lipschitz continuous* with $M = 1$

Then, the Lipschitz continuity of the objective function follows from that of the loss function. From theorem 1 clearly our objective function is a bounded from above, convex and Lipschitz continuous.

Theorem 2

Rademacher's Theorem states that; If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, then f is differentiable almost everywhere.

By this theorem, our objective function is differentiable almost everywhere since is Lipschitz continuous.

2. PARAMETER ESTIMATION AND ESTIMATOR PROPERTIES

2.1 ESTIMATION

For notational convenience let's denote the objective function as $\aleph(Y_t, \pi_{X_t}, \beta_\theta)$. An appealing method of estimation of the regression coefficients is to find the set of values of the coefficients which makes the residuals as small as possible. We define our estimates as those parameters that minimize the residuals dispersion. The parameters vector β_θ , is the solution to the minimization problem of the objective function.

$$\beta_\theta = \underbrace{\arg \min}_{\beta} \aleph(Y_t, \pi_{X_t}, \beta_\theta) \tag{8}$$

The $(r \times 1)$ vector β_θ is the population solution of the minimization problem which gives the least absolute residual error. Since to study an entire population is tedious we shall have to use a representative random sample $\{(Y_1, X_1), (Y_2, X_2), \dots, (Y_n, X_n)\}$. The sample version of the objective function will thus be given by;

$$\mathfrak{N}(\widehat{Y_t, \pi_{X_t, \beta_\theta}}) = n^{-1} \sum_{t=1}^n \vartheta_\theta(Y_t - \pi(X_t, \beta_\theta)) \quad (9)$$

The respective parameters vector estimate $\hat{\beta}_\theta$, is expressed as;

$$\hat{\beta}_\theta = \underbrace{\arg \min}_{\beta} \mathfrak{N}(\widehat{Y_t, \pi_{X_t, \beta_\theta}}) \quad (10)$$

Thus the conditional quantile function estimate is given by; $\widehat{Q}_\theta(Y_t | X_t) = \pi(X_t, \hat{\beta}_\theta) = \hat{\beta}'_\theta X_t$. Next, we define the InterQuantile AutoRegression Range at θ -level denoted by $IQARR_\theta$ as; $IQARR_\theta = \pi(X_t, \beta_\theta) - \pi(X_t, \beta_{1-\theta})$ and the estimate for the $IQARR_\theta$ is expressed as; $\widehat{IQARR}_\theta = \pi(X_t, \hat{\beta}_\theta) - \pi(X_t, \hat{\beta}_{1-\theta})$

From (1), we consider an AR (r)-GARCH (p, q) process which is a special case of QAR (r)-GARCH (p, q) which is mathematically expressed as;

$$Y_t = \pi(X_t, \beta) + \sigma(X_t, \alpha) e_t \quad (11)$$

Then manipulate e_t such that, $e_t = Z_t + q_\theta^e$, this makes the model be a quantile model of the form: $Y_t = \pi(X_t, \beta) + \sigma(X_t, \alpha) (Z_t + q_\theta^e)$ (12)

Where q_θ^e is the θ -quantile of e_t and Z_t is as defined previously and $e_t \sim iid(0,1)$, i.e. e_t has unit scale and mean zero. Substitution of Z_t with $e_t - q_\theta^e$ in the QAR (r)-GARCH (p, q) process at θ -quantile level and similarly for $1 - \theta$ -quantile level we get; $Y_t = \pi(X_t, \beta_\theta) + \sigma(X_t, \alpha) (e_t - q_\theta^e)$, and $Y_t = \pi(X_t, \beta_{1-\theta}) + \sigma(X_t, \alpha) (e_t - q_{1-\theta}^e)$ respectively. Taking the difference of these expressions and making the conditional scale function as the subject we have;

$$\sigma(X_t, \alpha) = \frac{\pi(X_t, \beta_\theta) - \pi(X_t, \beta_{1-\theta})}{q_\theta^e - q_{1-\theta}^e} = \frac{IQARR_\theta}{q_\theta^e - q_{1-\theta}^e} \quad (13)$$

Since the scale function can be expressed as a function of $IQARR_\theta$, we shall not estimate the parameters ω, α_θ and γ_θ . Therefore the conditional volatility function estimator is expressed as;

$$\hat{\sigma}(\tilde{X}_t, \alpha) = \frac{\pi(\tilde{X}_t, \hat{\beta}_{1-\theta}) - \pi(\tilde{X}_t, \hat{\beta}_\theta)}{q_{1-\theta}^e - q_\theta^e} = \frac{IQARR_\theta}{q_\theta^e - q_{1-\theta}^e} \tag{14}$$

The volatility function estimator $\hat{\sigma}(\tilde{X}_t, \alpha)$ will be given by substituting the respective parameters in the conditional quantile function quantities, the quantiles q_θ^e and $q_{1-\theta}^e$ will easily be obtained after assuming a distribution function for the nuisance term e_t .

2.2 ESTIMATOR PROPERTIES

Asymptotic properties of Estimator

To ensure that the estimator $\hat{\beta}_\theta$ is a good estimate of β_θ , we shall test for its asymptotic properties. We use the following assumptions as found in Mwita (2010).

CONDITION 1 (Consistency Assumptions)

Considering the QAR-GARCH process, the following assumptions are useful in providing surety for consistency of $\pi(\tilde{X}_t, \hat{\beta}_\theta)$.

A1. (F, F, P) is a complete probability space and $\{Z_t, X_t\}, t = 1, 2, 3, \dots$ are random vectors on this space

A2. The function $\pi(\tilde{X}_t, \beta_\theta): \mathbb{R}^{k_t} \times B \rightarrow \mathbb{R}$ is such that for each $\beta_\theta \in B$ a compact subset of $\mathfrak{R}^q, \pi(\tilde{X}_t, \hat{\beta}_\theta)$ is measurable with respect to the Borel set B^q and $\pi(\tilde{X}_t, \cdot)$ is continuous in B , a.s- $P, t = 1, 2, \dots$, for a given choice of explanatory variables $\{X_t\}$.

A3. (i) $E\left(\left[\vartheta_\theta\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right)\right]\right)$ exists and is finite for each β_θ in B ,
 (ii) $E\left(\left[\vartheta_\theta\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right)\right]\right)$ is continuous in β_θ and (iii) $\left\{\left[\vartheta_\theta\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right)\right]\right\}$ obeys the strong (weak) law of large numbers.

A4. $\left\{n^{-1}E\left\{\left[\vartheta_\theta\left(Y_t - \pi\left(\tilde{X}_t, \beta_\theta\right)\right)\right]\right\}\right\}$ has identifiable unique maximizer.

Theorem (Consistency)

Under assumptions A1– A4, $\hat{\beta}_\theta \rightarrow \beta_\theta$ as $n \rightarrow \infty$. Where $\hat{\beta}_\theta$ is as previously defined.

Proof

For the proof see White (1994, pg. 75) by using the loss function defined in (3).

CONDITION 2 (Asymptotic Normality)

To prove the asymptotic normality of $\hat{\beta}_\theta$, we introduce some extra notation. Let v_t be a $(r \times 1)$ vector of variables that determine the shape of the conditional distribution of $\epsilon_t = \sigma_t Z_t$. Associated with v_t is a set of parameters φ . Denote the density of ϵ_t , conditional on all the past information, as $h_t(\epsilon, \varphi, v_t)$, $\epsilon \in \mathfrak{R}$. Here, v_t includes conditional variance and φ , the vector of parameters that define a volatility model. Whenever the dependence on v_t and φ is not relevant, we will denote the conditional density of ϵ_t simply by $h_t(\epsilon)$. Let $u_t(\varphi, \beta_\theta, s)$ be an unconditional density of $s_t = (\epsilon_t, X_t, v_t)$. Finally, define the operators $\nabla \equiv \frac{\partial}{\partial \beta}$, $\nabla_i \equiv \frac{\partial}{\partial \beta_i}$, where β_i is the i th element of β , and $\nabla_i \pi_t(\beta) \equiv \nabla_i \pi(X_t, \beta)$ and $\nabla \pi_t(\beta) \equiv \nabla \pi_t(X_t, \beta)$.

The following assumptions are important for asymptotic normality.

B1. $\nabla_i \pi_t(\beta_\theta)$ is A -smooth (a function whose derivatives for all desired orders exist and are continuous within the given domain) with variables A_{it} and functions κ_i , $i = 1, 2, \dots, p$. In addition, $\max_i \kappa_i(\mathbf{d}) \leq \mathbf{d}$ for small enough.

B2. (i) $h_t(\epsilon)$ is Lipschitz continuous in ϵ uniformly in t . That is for $\epsilon_1, \epsilon_2 \in \epsilon$ and $l \in \mathbb{R}$ we have, $|h_t(\epsilon_1) - h_t(\epsilon_2)| \leq l|\epsilon_1 - \epsilon_2|$ (implying Lipschitz continuous) and $\forall \epsilon > 0$ there is a $\delta > 0$ s.t $|\epsilon_1 - \epsilon_2| < \delta \rightarrow |h_t(\epsilon_1) - h_t(\epsilon_2)| < \epsilon$ (implying h_t is uniformly continuous in ϵ .)

(ii) For each t and (ϵ, v) , $h_t(\epsilon, \varphi, v)$ is continuous in φ .

B3. For each t s , $u_t(\varphi, \beta_\theta, s)$ is continuous in (φ, β_θ) . (following from the continuity of β_θ)

B4. $\{\epsilon_t, X_t\}$ are α -mixing with parameter $\alpha(n)$, and there exist $\Delta < \infty$ and $r > 2$ such that $\alpha(n) \leq \Delta n^\omega$ for some $\omega < -\frac{2r}{r-2}$

B5. For some $r > 2$, $\nabla_i \pi_t(\beta)$ is uniformly r -dominated by functions a_{1t} .

B6. For all t and i , $E|\sup_\beta A_{it}|^r \Delta_1 < \infty$. There exist a measurable functions a_{2t} such that $|u_t| \leq a_{2t}$ and for all t , $\int a_{2t} dv < \infty$ and $\int a_{1t}^3 a_{2t} dv < \infty$.

B7. There exists a matrix A such that; $n^{-1} \sum_{t=a+1}^{a+n} E[\nabla \pi_t(\beta_\theta) \nabla' \pi_t(\beta_\theta)] \rightarrow A$

As $n \rightarrow \infty$ uniformly in a .

Theorem (Asymptotic Normality)

In consideration of our quantile autoregression model, if the estimator $\hat{\beta}_\theta$ is consistent and the axioms B1 – B7 hold, then we have;

$$\sqrt{n}A_n^{-\frac{1}{2}}D_n(\hat{\beta}_\theta - \beta_\theta) \xrightarrow{d} N(0,1)$$

Where;

$$A_n = \frac{\theta(1-\theta)}{n} \sum_{t=1}^n E[\nabla\pi_t(\beta_\theta)\nabla'\pi_t(\beta_\theta)], D_n = \frac{1}{n} \sum_{t=1}^n E[h_t(0)\nabla\pi_t(\beta_\theta)\nabla'\pi_t(\beta_\theta)] \text{ and}$$

$$\hat{\beta}_\theta = \underset{\beta}{\operatorname{argmin}} n^{-1} \left[\sum_{t \in \{t: Y_t \leq \beta'_\theta X_t\}} (\theta - 1) |Y_t - \beta'_\theta X_t| + \sum_{t \in \{t: Y_t > \beta'_\theta X_t\}} \theta |Y_t - \beta'_\theta X_t| \right]$$

Proof

To proof that the estimator is asymptotically normal we substitute the function $[\theta - I(x < 0)]$ in place of the function $\operatorname{sign}(x) = 2 \left[\frac{1}{2} - I_{\{x \leq 0\}} \right]$ in Weiss (1991) theorem 3.

If the conditions (A1-A4) and (B1-B7) are satisfied then our estimate is consistent and also asymptotically normal. Since our scale function is defined in terms of the interquantile autoregressive range then it is also consistent and asymptotically normal.

3. SIMULATION STUDY

A small simulation study was done for our model an AR (1) – GARCH (1, 1) to reinforce the theoretical results obtained earlier for samples of size $n=500$, $n=700$ and $n=1000$. The error term e_t is assumed to be independent and identically distributed following standard normal distribution. Figure 1, is an illustration of the AR (1) – GARCH (1, 1) and in Figure 2 we superimpose different QAR function estimates (at $\theta = 0.99, 0.95, 0.90, 0.75$ respectively) “lines” on the AR-GARCH process “points”.

The Interquantile Autoregression range function was estimated for all sample sizes at (a) $\theta=0.99$ and 0.01 (b) $\theta=0.95$ and 0.05 (c) $\theta=0.90$ and 0.10 and (d) $\theta=0.75$ and 0.25 . Figure 3, illustrates on the superimposing of symmetric quantile autoregression functions on the AR (1) – GARCH (1, 1) process. The computed volatility is

compared the true volatility and as seen in Figure 4, the estimated volatility follows the same pattern as the true volatility.

Quantiles exhibit a property of been robust to outliers. This is clearly illustrated in Figure 5 where its show how the parameters are dynamic from one quantile level to another. This makes the quantile autoregression model parameters adapt appropriately to capture outliers wherever they exist. To test the performance of the volatility estimator, we use its Mean Absolute Proportionate Error (MAPE).

$$MAPE(\hat{\sigma}(X_t, \alpha)) = \frac{1}{n} \sum_{t=1}^n \left| \frac{\sigma(X_t, \alpha) - \hat{\sigma}(X_t, \alpha)}{\sigma(X_t, \alpha)} \right|$$

As tabulated on Table 1, it is seen that the volatility estimator converges to the true volatility as the sample size increases. When the sample size is increased the MAPE tends to zero. That is; $\hat{\sigma}(X_t, \alpha) \rightarrow \sigma(X_t, \alpha)$ as $n \rightarrow \infty$. This property proofs the consistency of our estimates.

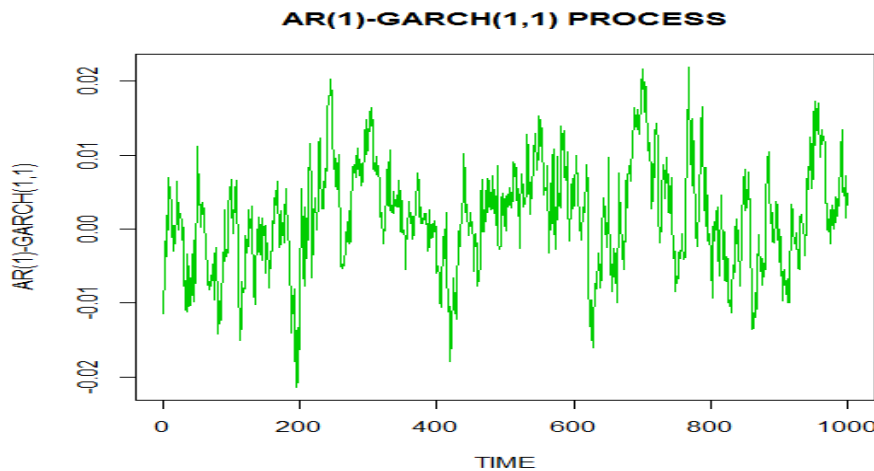


Figure 1: The simulated AR (1) – GARCH (1, 1) process

From the plot, we can see that there is; volatility clustering, low auto autocorrelation, high autocorrelation which are the stylistic features of financial data. Thus the AR (1)-GARCH (1, 1) can model financial phenomenon.

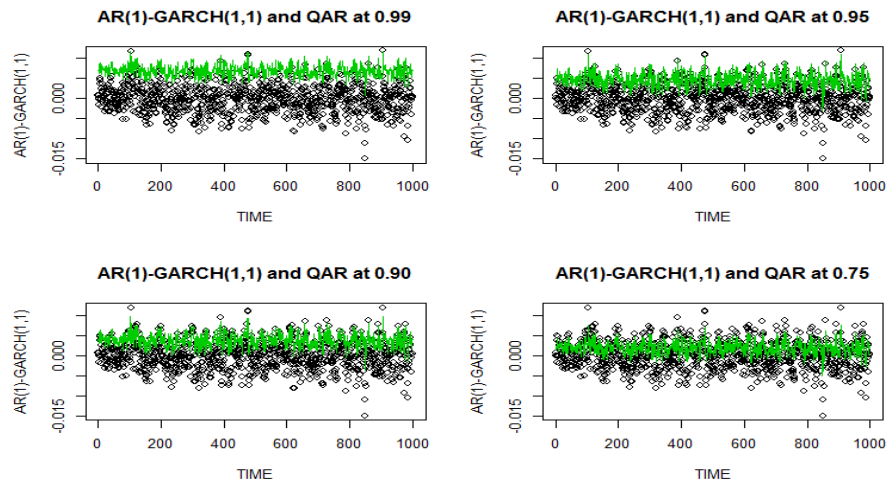


Figure 2: AR (1)-GARCH (1, 1) process (points) with Estimated QARF (green lines) superimposed.

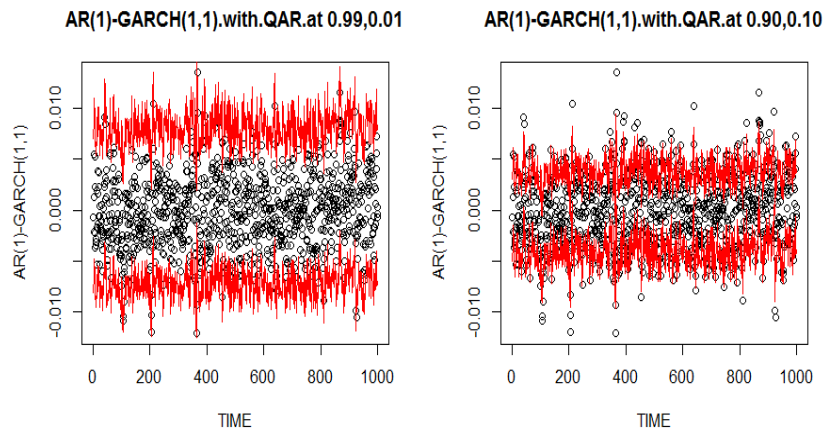


Figure 3: AR (1)-GARCH (1, 1) process (points) with Estimated QARF (red lines) “upper and lower quantiles” superimposed

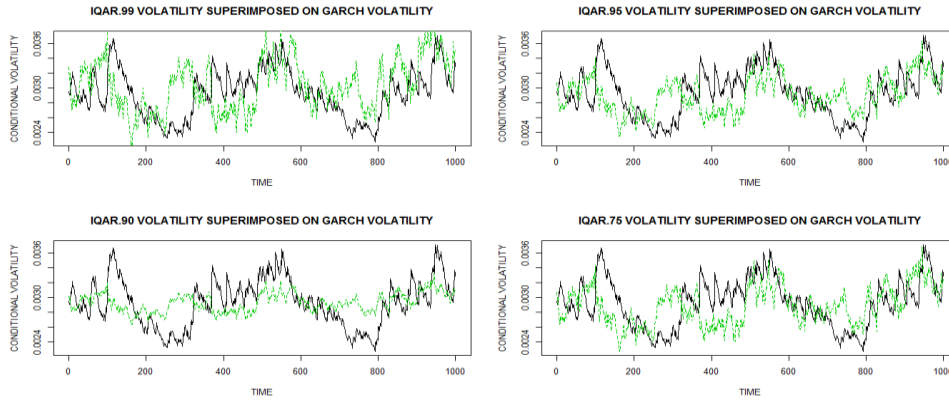


Figure 4: The Estimated Volatility “green dotted line” at different Quantiles superimposed on the True Volatility (GARCH generated) for a sample of size 1000

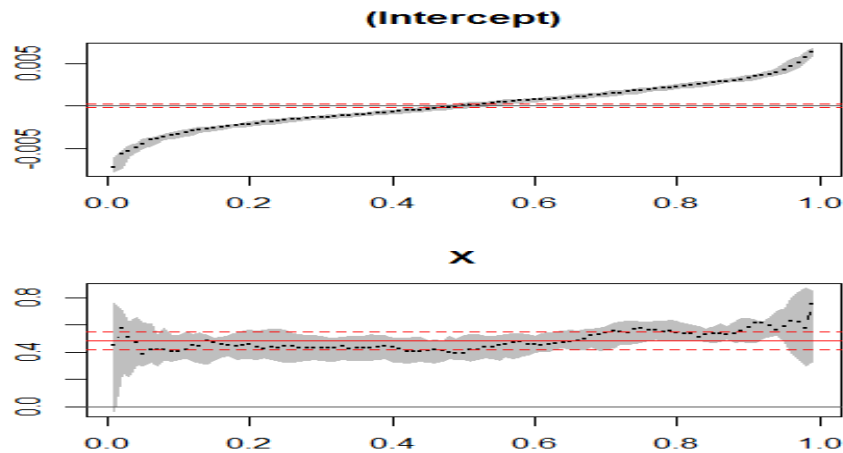


Figure 5: A plot of Quantile Autoregression Function parameter Estimates for different quantiles

Table 1: MAPE table with Increasing Sample sizes under different quantile levels

THETA θ -Quantile level	SAMPLE SIZE		
	N=500	N=700	N=1000
0.99	0.08879089	0.07060485	0.05853213
0.95	0.08214568	0.07584826	0.06694122
0.90	0.09705549	0.0843745	0.06034376
0.75	0.1164644	0.07893194	0.05859721

4. CONCLUSION

From the research in this paper we have come up with a quantile autoregression model, QAR – GARCH used in estimation of market risk volatility. The model being based on the interquantile autoregression range framework we have found that the method is dynamic and robust to outliers. The theoretical results obtained in this paper agree with the simulated results that the volatility estimator is consistent.

Further investigations can be done on cases where the error distribution is asymmetric. The method can also be improved by incorporating cases of data censoring. Future extension of the methodology can be done on Bayesian quantile autoregression and developing the methodology for forecasting.

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