umma university
Fostering knowledge and innovation

P. O. Box 713—01100 KAJIADO, KENYA TEL: 0739 969020, 0739 969021
Website: http://www.umma.ac.ke Email: vc@umma.ac.ke

REF: UU/AC/ROI/VOL.3/22 2nd November 2018

Dr. Stanley Rotich
Department of Mathematics, Statistics and Actuarial Science
School of Pure and Applied Sciences
Machakos University
P.O BOX 136-90100, Machakos

Dear Sir/Madam,

RE: INVITATION TO PRECONFERENCE TRAINING AND AN INTERNATIONAL CONFERENCE ON SCIENCE, TECHNOLOGY AND INNOVATION FOR SUSTAINABLE DEVELOPMENT IN DRYLAND ENVIRONMENTS AND

Umma University in Kajiado County, South Eastern Kenya University (SEKU) in Kitui County, Lukenya University in Makueni County and Machakos University in Machakos County and together with other partners are jointly organizing an international Conference entitled *"Science, Technology and Innovation for Sustainable Development in Dryland Environments"* to be held on 19th-23rd November 2018. The theme of the conference is *"Harnessing Dryland Natural Resources for Sustainable Livelihoods in the Era of Climate Change*". The conference will be two-phased with a two day pre-conference training workshop on 19th-20th November 2018 at SEKU and the actual conference on 21st-23rd November 2018 at Umma University. The conference will provide an excellent platform for the academia from around the world to engage with the industry, innovators, policy makers, value chain developers, farmers, and service providers among others so that higher education in Africa contribute to the solutions to the problems of natural resources governance in the era of climate change.

We are therefore pleased to invite you to attend the pre-conference training workshop at SEKU main Campus in Kitui on 19th-20th November 2018 and the Actual Conference at Umma University Main Campus, Kajiado on 21st to 23rd November 2018. Please note that you will be responsible for your travel and accommodation arrangements and conference registration fee.

Yours Sincerely,

DR. ALI ADAN ALI
FOR THE: VICE-CHANCELLOR

SUBORBITAL GRAPHS CORRESPONDING TO PERMUTATION REPRESENTATIONS OF $PGL(2,q)$

S.K. Rotich

Department of Mathematics, Statistics and Actuarial Science
Machakos University
P.O. Box 136,
Machakos-Kenya
Corresponding author
e-mail: starotich@gmail.com

ABSTRACT

In this paper we construct Suborbital graphs for $PGL(2,q)$ acting on the cosets of C_{q-1} and analyze their properties. We established that the number of self paired suborbits is $q+2$ and the paired suborbits are 2. Also suborbital graphs corresponding to suborbits whose elements intersect $\{0,\infty\}$ at a singleton have been shown to be of girth 3. Suborbital graph corresponding to the suborbit containing $(0,\infty)$ is found to be of girth 0. Finally suborbital graph corresponding to suborbit with representative of the form $(1,\beta^i)$ is shown to be of girth 4.

Keywords: *Suborbits, Suborbitals, Cosets*

1.0 Introduction

Let G be a transitive permutation group acting on a set X. Then G acts on $X\times X$ by $g(x,y)=(gx,gy)$, $g\in G, x,y\in X$. If $O\subseteq X\times X$ is a G-Orbit on $X\times X$, then for a fixed $x\in X$, $\Delta=\{y\in X|(x,y)\in O\}$ is a G_x-Orbit. Conversely, if $\Delta\subset X$ is a G_x-Orbit, then $O=\{(gx,gy)|g\in G, y\in\Delta\}$ is a G-Orbit on $X\times X$. We say Δ corresponds to O. The G-orbits on $X\times X$ are called suborbitals. Let $O_i\subseteq X\times X, i=0,1,\ldots,r-1$ be a suborbital. Then we form a graph Γ_i, by taking X as the set of vertices of Γ_i and by including a directed edge from x to y $(x,y\in X)$ if and only if $(x,y)\in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now $O_i^*=\{(x,y)|(y,x)\in O_i\}$ is also a G-Orbit. Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^*. Let the suborbits Δ_i $(i=0,1,\ldots,r-1)$ correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and directed if Δ_i is not self-paired. (Sims [3])

1.1 Theorem

Because H is the stabilizer of an ordered pair, we deduce the following from Theorem 1.1;

2.1 Corollary

The action of G on the cosets of H is equivalent to its action on the ordered 2- element subsets from the projective line.

Proof

Using Theorem 1.1 the result is immediate. ■

From now henceforth we shall be working on the action of G on the ordered pairs from the projective line where $PG(1,q)=\{\infty,0,1,\beta,\beta^2,\ldots\ldots\ldots\ldots,\beta^{q-2}\}$. Let H be the stabilizer of $(\infty,0)$ and β be a primitive element of $GF(q)$, then

$$H\equiv<\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}>\equiv(0)(\infty)(1\ \beta\ \beta^2\ \ldots\ldots\ldots\ldots\ \beta^{q-2}).$$

So the suborbits of G on ordered pairs from $PG(1,q)$ are;

$$Orb_{G_{(\infty,0)}}(\infty,0)=\{(\infty,0)\}=\Delta_0$$

$$Orb_{G_{(\infty,0)}}(0,\infty)=\{(0,\infty)\}=\Delta_1$$

$$Orb_{G_{(\infty,0)}}(\infty,1)=\{(\infty,1),(\infty,\beta),(\infty,\beta^2),\ldots\ldots,(\infty,\beta^{q-2})\}=\Delta_2$$

$$Orb_{G_{(\infty,0)}}(1,\infty)=\{(1,\infty),(\beta,\infty),(\beta^2,\infty),\ldots\ldots,(\beta^{q-2},\infty)\}=\Delta_3$$

$$Orb_{G_{(\infty,0)}}(0,1)=\{(0,1),(0,\beta),(0,\beta^2),\ldots\ldots,(0,\beta^{q-2})\}=\Delta_4$$

$$Orb_{G_{(\infty,0)}}(1,0)=\{(1,0),(\beta,0),(\beta^2,0),\ldots\ldots,(\beta^{q-2},0)\}=\Delta_5$$

$$Orb_{G_{(\infty,0)}}(1,\beta)=\{(1,\beta),(\beta,\beta^2),(\beta^2,\beta^3),\ldots\ldots,(\beta^{q-2},1)\}=\Delta_6$$

$$Orb_{G_{(\infty,0)}}(1,\beta^2)=\{(1,\beta^2),(\beta,\beta^3),(\beta^2,\beta^4),\ldots\ldots,(\beta^{q-2},\beta)\}=\Delta_7$$

....

$$Orb_{G_{(\infty,0)}}(1,\beta^{q-2})=\{(1,\beta^{q-2}),(\beta,1),(\beta^2,\beta),\ldots\ldots,(\beta^{q-2},\beta^{q-3})\}=\Delta_{q+3}$$

So the subdegrees are as shown in Table 2.2 below

$2(q-3)(\frac{q(q+1)}{2})$ to the formula. If g is an elliptic element of order 2 then g^2 fixes $q(q+1)$ elements. In total we have $(\frac{q(q-1)}{2})$ elliptic elements of order 2 hence contribute $q(q+1)(\frac{q(q-1)}{2})$ to the formula. Now applying Theorem 6.1.6 we have;

$$n_{\pi}=\frac{1}{|G|}\left[q(q+1)+0+2(q-3)\left(\frac{q(q+1)}{2}\right)+q(q+1)\left(\frac{q(q+1)}{2}\right)\right.$$
$$\left.+q(q+1)\left(\frac{q(q-1)}{2}\right)\right]=q+2. \blacksquare$$

Since the rank of G on the cosets of H is $q+4$, the number of paired suborbits are 2.

2.4 Theorem

When G acts on the set of ordered pairs from $PG(1,q)$, then Δ_3 is paired with Δ_4.

Proof

Let $g \in G$, where $g=\begin{pmatrix}0 & -1\\ -1 & 1\end{pmatrix}$, then g maps $(1,\infty) \in \Delta_3$ to $(\infty,0)$ and $(\infty,0)$ to $(0,-1) \in \Delta_4$. Therefore by the introduction Δ_3 and Δ_4 are paired. ■

3.0 Suborbital graphs Γ of G corresponding to its suborbits when G acts on the cosets of $H = C_{q-1}$

We recall from Section 2.0 that there are $q+2$ suborbits of length $q-1$ and 2 suborbits of length 1.

3.1 Suborbital graph corresponding to suborbit of G formed by pairs of the form (β^i,∞)

Since G is doubly transitive on the $PG(1,q)$; given a pair (v,h) $v \neq h$ v, and

$$\begin{pmatrix}1 & h\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \infty = d.$$

Hence $c = (\beta^i + h)$ and $d = \infty$.

c) $h = \infty$, so we choose $g \in G$ to be $\begin{pmatrix}v & 1\\1 & 0\end{pmatrix}$, hence we have,

$$g(\beta^i, \infty) = (c, d)$$

$$\begin{pmatrix}v & 1\\1 & 0\end{pmatrix}\begin{pmatrix}\beta^i 1\\1\,0\end{pmatrix} = \begin{pmatrix}v\beta^i + 1 & v\\\beta^i & 1\end{pmatrix}.$$

Thus $d = v$ and $c = (v\beta^i + 1)\beta^{-i}$. ■

3.2 Suborbital graph corresponding to suborbit of G formed by pairs of the form $(\beta^i, 0)$

3.2.1 Theorem

For each of the following cases $((v,h),(c,d))$ is an edge in $\Gamma_{(\beta^i,0)}$.

a) $v, h \neq \infty$, $d = h$ and $c = (v\beta^i + h)(\beta^i + 1)^{-1}$

b) $d = h = \infty$ and $c = (v\beta^i + 1)\beta^{-i}$

c) $v = \infty$, $d = h$ and $c = \beta^i + h$

Proof

a) Using the same approach as Theorem 6.3.1.1, when $v, h \neq \infty$, we let $g \in G$ be $\begin{pmatrix}v & h\\1 & 1\end{pmatrix}$.

So

$$g(\beta^i, 0) = (c, d).$$

Hence:

$$\begin{pmatrix}v & h\\1 & 1\end{pmatrix}\begin{pmatrix}\beta^i\\1\end{pmatrix} = (v\beta^i + h)(\beta^i + 1)^{-1} = c$$

and

Thus $c = h$, $d = (v\beta^i + h)(\beta^i + 1)^{-1}$.

b) For $v = \infty$, so taking $g \in G$ to be $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, so we have;

$$g(0, \beta^i) = (c, d)$$

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \beta^i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} h & \beta^i + h \\ 1 & 1 \end{pmatrix}.$$

So $c = h$ and $d = \beta^i + h$.

c) Finally for $h = \infty$, then we have,

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta^i \\ 1 & 1 \end{pmatrix} = (\infty, (v\beta^i + 1)\beta^{-i}) = (c, d).$$

Thus $c = \infty$ and $d = (v\beta^i + 1)\beta^{-i}$. ■

3.4 Suborbital graph corresponding to suborbit of G formed by pairs of the form (∞, β^i)

3.4.1 Theorem

For each of the following cases$((v, h), (c, d))$ is an edge in $\Gamma_{(\infty, \beta^i)}$.

a) $v, h \neq \infty$, $c = v$ and $d = (v\beta^i + h)(\beta^i + 1)^{-1}$

b) $v = c$ and $d = (v\beta^i + 1)\beta^{-i}$

c) $v = c = \infty$, and $d = \beta^i + h$.

Proof

a) $v, h \neq \infty$, so we take $g \in G$ to be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$; so

$$g(\infty, \beta^i) = (c, d)$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta^i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} v & v\beta^i + h \\ 1 & \beta^i + 1 \end{pmatrix} = (v, (v\beta^i + h)(\beta^i + 1)^{-1}).$$

$$= ((v+h)(2)^{-1}, (v\beta^i + h)(\beta^i+1)^{-1})$$

Thus $c = (v+h)(2)^{-1}$ $d = (v\beta^i + h)(\beta^i+1)^{-1}$.

b) Since $h = \infty$, In this case we take $g \in G$ to be $\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}$, so we have;

$$g(1, \beta^i) = (c, d)$$

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \beta^i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} v+1 & \beta^i+1 \\ 1 & \beta^i \end{pmatrix}$$

So $c = v + 1$ and $d = (\beta^i + 1)\beta^{-i}$

c) For $v = \infty$, then we have,

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta^i \\ 1 & 1 \end{pmatrix} = ((1+h), (\beta^i + h)) = (c, d)$$

Hence $c = 1 + h$ and $d = \beta^i + h$ ■

3.6 Suborbital graph corresponding to suborbit of G formed by the pairs $(0, \infty)$

3.6.1 Theorem

For each of the following cases $((v, h), (c, d))$ is an edge in $\Gamma_{(0,\infty)}$.

a) $v, h \neq \infty$, $c = h$ and $d = \text{v}$

b) $c = h$ and $d = \text{v} = \infty$

c) $c = h = \infty$ and $d = v$

Proof

a) Since $v, h \neq \infty$, then we take $g \in G$ to be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$. So we have;

$$g(0, \infty\ ) = (c, d).$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h & v \\ 1 & 1 \end{pmatrix} = (h, \text{v})$$

Thus $c = h$ $d = \text{v}$.

The suborbital graphs $\Gamma_{(1,\beta^i)}$ is of girth 4.

Proof

Since $(1,\boldsymbol{\beta}),(\boldsymbol{\beta},1) \in \Delta_6$, in $\Gamma_{(1,\beta^i)}$, $(\infty, 0)$is adjacent to $(1,\boldsymbol{\beta})$ and $(\boldsymbol{\beta},1)$. By Lemma 6.1.8 and Theorem 3.5.1 and taking (v,h) to be $(1,\boldsymbol{\beta})$and β^i to be -1 we find that $((1,\boldsymbol{\beta}),(0,\infty))$ is an edge. Also applying the same Theorem and taking (v,h) to be $(0,\infty)$ we find that $((0,\infty),(\boldsymbol{\beta},1))$ is an edge in $\Gamma_{(1,\beta^i)}$ giving us girth 4.

4.4 Theorem

The number of connected components for $\Gamma_{(0,\infty)}$ is $\frac{q(q+1)}{2}$.

Proof

The number of ordered pairs is $\binom{q+1}{2}$. Each connected components has two ordered pairs. Hence the results follow.

REFERENCES

1. Cameroon, P. J. (1975). Suborbits in transitive permutation groups. *Proceedings of the NATO Advanced Study Institute on Combinatorics*, Breukelen, Netherlands.

2. Rose, J. S. (1978). *A Course on group theory*. Cambridge University Press, Cambridge.

3. Sims, C. C. (1967). Graphs and finite permutation groups. *Mathematics Zeitschrift* 95: 76 – 86.

4. Wielandt, H. (1964). *Finite permutation groups*. Academic Press, New York.