

Pricing Interest Rate Caps and Floors under the Pearson-Sun Interest Rate Model

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Abstract

Interest rate derivatives are largely used to manage interest rate risk. For this reason, the accuracy of their pricing is very important as mispricing can result in huge financial losses. In this study, the Pearson-Sun model, an extension of the CIR model, was used to price interest rate caps and floors. In the pricing process, the prices of zero-coupon bonds and European options on zero-coupon bonds were derived. These were then used to obtain the prices of caps and floors. The parameters of the Pearson-Sun model were estimated using maximum likelihood method on a daily term structure time series data. The results of the study showed that the CIR model is rejected in favour of the Pearson-Sun model. This implies that it would provide a better pricing accuracy as

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compared to the CIR model and would provide better prices to interest rate derivatives. The prices of caps and floors were simulated using the estimated parameters under the Pearson-Sun model.

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1 Introduction

Interest rate derivatives are popularly used to manage interest rate risk. The popularity of these derivatives has made their pricing a major concern and yet a challenging task. This has therefore attracted the attention of many researchers in recent times [1]. Mispricing of these derivatives can result in mismanagement of interest rate risk, which can lead to huge losses. To price interest rate derivatives, term structure models are used. These models have gone through tremendous development over the years.

In financial asset pricing, the Black-Scholes model proposed by [2] was a major breakthrough. However, the model was not realistic with real markets as it considered a constant interest rate. Stochastic processes were then employed to describe interest rates and was first introduced by [8]. This was later extended by Ho and Lee [6].

The initial models were not characterised by mean reversion and diffusion [6]. Because of this, they could not simulate the real world rates very well. [11] proposed a model which captured these two characteristics. But this model also admitted negative rates, which was undesirable. This was corrected by the CIR model proposed by [3].

Even though the CIR model is widely preferred and used, it does not always describe interest rates movements in the observed market. The CIR model does not always provide satisfactory pricing results and does not always account for the shapes of term structure observed in the bond markets [7]. Therefore, over the years, a number of extensions of the CIR model have been constructed. Some of these include Hull and White model by [5] and the Pearson-Sun model by [9]. The Pearson-Sun model is an extension of the CIR model which preserves the drift term and introduces a constant to the CIR interest rate model.

The Pearson-Sun model makes the realisation of small values much more likely as compared to the CIR model [9]. That is, the density of the CIR model assigns less probability to values near zero than that of Pearson-Sun model. As

some economies have had very low interest rates over the years, the CIR model may not be able to simulate such rates because of this weakness.

In [9], the price of a zero-coupon bond was obtained by re-writing the Pearson-Sun model into a CIR model. In this study, the pricing formula of zero-coupon bonds and the pricing formula of European options written on them under the Pearson-Sun model are derived. These two derivations are the key contributions of the paper. These prices were used to obtain the prices of caps and floors under the model. Again, the Pearson-Sun model is compared with the CIR model on a term-structure time series data. The parameters of the model are estimated using maximum likelihood method. The results of the study show that the Pearson-Sun model outperforms the CIR model and hence will provide a better pricing accuracy for interest rate dependent assets.

2 Important Concepts

In this section, some basic concepts relevant to the study are presented. The section contains term structure of interest rates and some standard results for the prices of zero-coupon bonds, caps and floors.

2.1 Term structure of interest rates

A term structure of interest rates define the evolution of interest rates over time. The major term structure model considered in this study is the model proposed by [9], and the benchmark model used is the CIR interest rate model.

The Pearson-Sun model is described by the following stochastic differential equation

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{(r(t) - r_b)}d\mathcal{B}(t), \quad r(0) \geq 0, \quad t \geq 0 \quad (1)$$

where μ is the long term mean, α is rate of mean reversion to the long term mean, σ is the volatility, r_b is the lower bound introduced and $\mathcal{B}(t)$ is a standard Brownian motion. The Pearson-Sun model retains the drift term of the CIR model given below and introduces a bound to the interest rates.

The benchmark model, CIR model, is given by the equation

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\mathcal{B}(t), \quad r(0) \geq 0, \quad t \geq 0 \quad (2)$$

where μ , α and σ are the parameters of the model and are defined as in the Pearson-Sun model and $\mathcal{B}(t)$ is a standard Brownian motion.

2.2 Zero-coupon bond prices

The time t price of a zero-coupon bond under the risk-neutral framework with maturity T is given by

$$P(t, T) = \mathbb{E}_Q \left[e^{-\int_t^T r(u)du} \mid \mathcal{F}(t) \right]. \quad (3)$$

A standard result of equation (3) is given in [10] as

$$P(t, T) = e^{A(t, T) - B(t, T)r(t)}$$

where $A(t, T)$ and $B(t, T)$ are deterministic functions.

2.3 Price of a Cap and Floor

The price of a cap and a floor are considered as a portfolio of European put options on zero-coupon bonds and European call options, respectively [4].

Consider a cap/floor maturing at time T with strike rate K and nominal amount N . Also let the reset times for the cap be $t_i, i = 1 \dots, n$ and payment times $t_i, i = 2, \dots, n$. Let τ_i be the time fraction between t_{i-1} and t_i .

The price of a cap under the Pearson-Sun model at time t is given by

$$Cap(t) = N \sum_{i=1}^n [(1 + K\tau_i)ZBP(t, t_{i-1}, t_i, \bar{K}_i)] \quad (4)$$

where $ZBP(\cdot)$ is the price of a European put option on zero-coupon bond. The price of the floor at time t is given by

$$floor(t) = N \sum_{i=1}^n [(1 + K\tau_i)ZBC(t, t_{i-1}, t_i, \bar{K}_i)] \quad (5)$$

where $ZBC(\cdot)$ is a European call option on zero-coupon bond and $\bar{K}_i = \frac{1}{(1+K\tau_i)}$.

3 Mathematical Results

In this section, the main mathematical results of the study under the Pearson-Sun model are presented.

3.1 Density function

The density function of the model is used to estimate the parameters of the model via maximum likelihood method.

Proposition 3.1. *Given that the dynamics of the short-rate is defined by the Pearson-Sun interest rate model given by equation (1). The probability density function at time s conditional on its value at the current time t is given by*

$$f_r(r(s), s; r(t), t) = ce^{-u-v} \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2\sqrt{uv}) \quad (6)$$

where

$$\begin{aligned} c &= \frac{2\alpha}{\sigma^2(1 - e^{-\alpha(s-t)})} \\ u &= c[r(t) - r_b]e^{-\alpha(s-t)} \\ v &= c[r(s) - r_b] \\ q &= \frac{2\alpha(\mu - r_b)}{\sigma^2} - 1 \end{aligned} \quad (7)$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q .

Proof. To proof the above proposition, it should be noted that the Pearson-Sun model given by equation (1) could be written as

$$d(x(t)) = \alpha([\mu - r_b] - x(t))dt + \sigma\sqrt{x(t)}d\mathcal{B}(t) \quad (8)$$

where $x(t) = r(t) - r_b$.

Equation (8) can be seen as a CIR interest rate model with long-run mean $\mu - r_b$. Its general conditional density function is given in [3]. Using the technique of transformation of random variables, the distribution function given in the proposition is obtained. \square

3.2 Price of Zero-Coupon Bonds

A zero-coupon bond pays no dividend during the life-time of the bond with a terminal payoff of 1. The following proposition presents the price of a zero-coupon bond under the Pearson-Sun model.

Proposition 3.2. *Consider that the dynamics of the short-rate is given by the Pearson-Sun model. The time t price of a zero-coupon bond maturing at time T , denoted $P(t, T)$, is given by*

$$P(t, T) = e^{\Lambda(t, T) - B(t, T)r(t)} \quad (9)$$

where

$$\begin{aligned}
 B(t, T) &= \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + \alpha)(e^{\gamma(T-t)} - 1) + 2\gamma} \\
 \Lambda(t, T) &= A(t, T) + C(t, T) \\
 A(t, T) &= \frac{2\alpha\mu}{\sigma^2} \ln \left(\frac{2\gamma e^{\frac{(\alpha+\gamma)}{2}(T-t)}}{(\alpha + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma} \right) \\
 C(t, T) &= 2r_b \left[\frac{1}{(\alpha - \gamma)} \ln \left\{ \frac{(2\gamma)^l e^{\frac{(\alpha+\gamma)}{2}(T-t)}}{((\alpha + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma)^{\frac{2\alpha}{\alpha+\gamma}}} \right\} \right. \\
 &\quad \left. + \frac{1}{(\alpha + \gamma)} \left(\frac{1}{2\gamma} - \frac{1}{((\alpha + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma)} \right) \right] \\
 l &= \frac{2\alpha}{(\alpha + \gamma)} \\
 \gamma &= \sqrt{\alpha^2 + 2\sigma^2}.
 \end{aligned}$$

Proof. Given that interest rate process is given by equation (1), the bond price at time t with maturity at time T is defined by equation (3). Multiplying both sides of equation (3) by $e^{-\int_0^t r(u)du}$, we obtain a Martingale given as

$$P(t, T)e^{-\int_0^t r(u)du} = \mathbb{E}_Q \left[e^{-\int_0^T r(u)du} \mid \mathcal{F}(t) \right]. \tag{10}$$

For simplicity, let $P(t, T) = P$ and $r(t) = r$. Using Ito lemma, the differential of the LHS of equation (10) is given as

$$d \left(P e^{-\int_0^t r(u)du} \right) = -r P e^{-\int_0^t r(u)du} dt + e^{-\int_0^t r(u)du} \left[P_t dt + P_r dr + \frac{1}{2} P_{rr} (dr)^2 \right] \tag{11}$$

where

$$P_t = \frac{\partial P}{\partial t}, \quad P_r = \frac{\partial P}{\partial r}, \quad P_{rr} = \frac{\partial^2 P}{\partial r^2}.$$

Since dr is given by equation (1), $(dr)^2 = \sigma^2(r - r_b)dt$. Substituting into (11) and given the fact that (10) is a Martingale, we obtain the partial differential equation

$$\left. \begin{aligned}
 P_t + \alpha(\mu - r)P_r + \frac{1}{2}P_{rr}\sigma^2(r - r_b) - rP &= 0 \\
 \text{with terminal condition } P(T, T) = 1, \quad r &\geq 0
 \end{aligned} \right\} \tag{12}$$

Equation (12) has a standard closed-form solution of the form

$$P(t, T) = e^{\Lambda(t, T) - B(t, T)r(t)} \tag{13}$$

where $\Lambda(t, T)$ and $B(t, T)$ are deterministic functions. Differentiating $P(t, T)$ and substituting into equation (12) gives

$$\left[(-B_t + \alpha B + \frac{1}{2}B^2\sigma^2 - 1)r + (\Lambda_t - \alpha\mu B - \frac{1}{2}r_b B^2\sigma^2) \right] P = 0.$$

Now, the solutions to the following differential equations below give B and Λ .

$$\left. \begin{aligned} -B_t + \alpha B + \frac{1}{2}B^2\sigma^2 - 1 &= 0 \\ B(T, T) &= 0 \end{aligned} \right\} \tag{14}$$

$$\left. \begin{aligned} \Lambda_t - \alpha\mu B - \frac{1}{2}r_b B^2\sigma^2 &= 0 \\ \Lambda(T, T) &= 0. \end{aligned} \right\} \tag{15}$$

Equation (14) is a Ricatti equation. The solution to (14) is as given in the proposition. The solution to equation (15) is obtained by solving the following equation

$$\Lambda(t, T) = -\alpha\mu \int_t^T B ds - \frac{1}{2}\sigma^2 r_b \int_t^T B^2 ds. \tag{16}$$

Substituting the expression for B into equation (16), with change of limits, we obtain

$$\begin{aligned} \Lambda(t, T) &= \frac{2\alpha\mu}{\gamma} \int_{e^{\gamma(T-t)}}^1 \left(\frac{x-1}{(\alpha+\gamma)(x-1)+2\gamma} \right) \frac{dx}{x} + \frac{2\sigma^2 r_b}{\gamma} \int_{e^{\gamma(T-t)}}^1 \left(\frac{x-1}{(\alpha+\gamma)(x-1)+2\gamma} \right)^2 \frac{dx}{x} \\ &= A(t, T) + C(t, T). \end{aligned} \tag{17}$$

The solutions to $A(t, T)$ and $C(t, T)$ are as given in the proposition. □

3.3 Price of European Options on Zero-Coupon Bond

The price of European call option is given in the following proposition.

Proposition 3.3. *Given that interest rates follow the Pearson-Sun term structure model given in equation (1) above, for $0 \leq t \leq s \leq T$, the time t price of a European call option expiring at time s with exercise price K , denoted $ZBC(t, s)$, on a zero-coupon bond maturing at time T , is given by*

$$ZBC(t, s) = P(t, T)\chi^2(d, y_1; w_1) - KP(t, s)\chi^2(d, y_2; w_2) \tag{18}$$

where $\chi^2(d, y; w)$ is the cumulative distribution of a non-central chi-squared

distribution with d degrees of freedom and non-centrality parameter y and

$$\begin{aligned} d &= \frac{4\alpha(\mu - r_b)}{\sigma^2} \\ y_1 &= \frac{2\phi r e^{\lambda(s-t)}}{\phi + \psi + B(s, T)} \\ w_1 &= 2\eta [\phi + \psi + B(s, T)] \\ y_2 &= \frac{2\phi r e^{\lambda(s-t)}}{\phi + \psi} \\ w_2 &= 2\eta [\phi + \psi] \\ \lambda &= \sqrt{\alpha^2 + 2\sigma^2} \\ \phi &= \frac{2\lambda}{\sigma^2 (e^{\lambda(s-t)} - 1)} \\ \psi &= \frac{\alpha + \lambda}{\sigma^2} \\ \eta &= \frac{\Lambda(s, T) - \ln K}{B(s, T)} - r_b. \end{aligned}$$

Proof. Let the maturity date of the European call option on the zero coupon bond to be s with strike price K and maturity of the zero coupon bond be T . The price at time t of the European call option given $r(t)$ is defined as

$$ZBC(t, s) = \mathbb{E}_Q \left[e^{-\int_t^s r(u) du} (P(s, T) - K)^+ \mid r(t) \right]. \quad (19)$$

The option will only be exercised if $P(s, T) > K$. But $P(s, T) = e^{\Lambda(s, T) - B(s, T)r(s)}$.

$$\Rightarrow e^{\Lambda(s, T) - B(s, T)r(s)} > K \iff r(s) < \frac{\Lambda(s, T) - \ln K}{B(s, T)} = \eta \Rightarrow r(s) < \eta.$$

Now, define an indicator function by

$$\mathbb{I}_{r(s) < \eta} = \begin{cases} 1 & \text{if } r(s) < \eta \\ 0 & \text{otherwise.} \end{cases}$$

And we have

$$ZBC(t, s) = \mathbb{E}_Q \left[e^{-\int_t^s r(u) du} P(s, T) \mathbb{I}_{r(s) < \eta} \mid r(t) \right] - \mathbb{E}_Q \left[e^{-\int_t^s r(u) du} K \mathbb{I}_{r(s) < \eta} \mid r(t) \right]. \quad (20)$$

$$= \mathbb{E}_Q \left[e^{-\int_t^T r(u) du} \mathbb{I}_{r(s) < \eta} \mid r(t) \right] - \mathbb{E}_Q \left[e^{-\int_t^s r(u) du} K \mathbb{I}_{r(s) < \eta} \mid r(t) \right]. \quad (21)$$

Employing change of measure; let P_1 and P_2 be two measures, equivalent to \mathbb{Q} . Defining their Radon-Nikodym derivatives and substituting gives

$$\begin{aligned} ZBC(t, s) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T r(u)du} \mid r(t) \right] \mathbb{E}_{\mathbb{Q}} \left[\frac{dP_1}{d\mathbb{Q}} \mathbb{I}_{r(s) < \eta} \mid r(t) \right] \\ - \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^s r(u)du} \mid r(t) \right] \mathbb{E}_{\mathbb{Q}} \left[\frac{dP_2}{d\mathbb{Q}} K \mathbb{I}_{r(s) < \eta} \mid r(t) \right]. \end{aligned}$$

But

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T r(u)du} \mid r(t) \right] \quad \text{and} \quad P(t, s) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^s r(u)du} \mid r(t) \right].$$

Therefore,

$$ZBC(t, s) = P(t, T) \mathbb{E}_{\mathbb{Q}} \left[\frac{dP_1}{d\mathbb{Q}} \mathbb{I}_{r(s) < \eta} \mid r(t) \right] - P(t, s) \mathbb{E}_{\mathbb{Q}} \left[\frac{dP_2}{d\mathbb{Q}} K \mathbb{I}_{r(s) < \eta} \mid r(t) \right] \tag{22}$$

and

$$ZBC(t, s) = P(t, T) \mathbb{P}_{P_1} (r(s) < \eta \mid r(t)) - KP(t, s) \mathbb{P}_{P_2} (r(s) < \eta \mid r(t)). \tag{23}$$

To obtain the probabilities under P_1 and P_2 , we note that Under P_1 , using equation (8), where $x(t)$ is described by the CIR model, under P_1 gives

$$\mathbb{P}_{P_1} (r(s) < \eta \mid r(t)) = \mathbb{P}_{P_1} (x(s) < \eta - b \mid x(t))$$

and similarly, under P_2 , this gives

$$\mathbb{P}_{P_2} (r(s) < \eta \mid r(t)) = \mathbb{P}_{P_2} (x(s) < \eta - b \mid x(t)).$$

Equation (23) becomes

$$ZBC(t, s) = P(t, T, r) \mathbb{P}_{P_1} (x(s) < \eta - b \mid x(t)) - KP(t, s, r) \mathbb{P}_{P_2} (x(s) < \eta - b \mid x(t)). \tag{24}$$

Since $x(t)$ is described by the CIR model, the probabilities under both P_1 and P_2 are well known and can be found in [3]. With the necessary substitutions, the results in Proposition 3.3 are established. \square

Using the put-call parity for bond options defined by

$$ZBP(t, s) = ZBC(t, s) + KP(t, s) - P(t, T) \tag{25}$$

where $ZBC(t, s)$ is the price of a European call option at time t , $ZBP(t, s)$ is the price of a European put option at time t , K is the strike price, $P(t, s)$ and $P(t, T)$ are the price of the zero coupon bond [4], the corresponding price of a European put option on the zero-coupon bond can be obtained.

4 Numerical Results

A comparison of the Pearson-Sun model and the CIR model in terms of their ability to describe daily time series data on yields is shown.

4.1 Data Description

The data used in this study consist of daily 3 month term structure data obtained from the bank of Canada. The data consist of 1972 daily observations ranging from January 2010 to November 2017. Figure 1 shows the plot of the data. It can be observed that the rates are fairly steady over the period with steep increase and decrease at specific short time periods. From Table 1, it

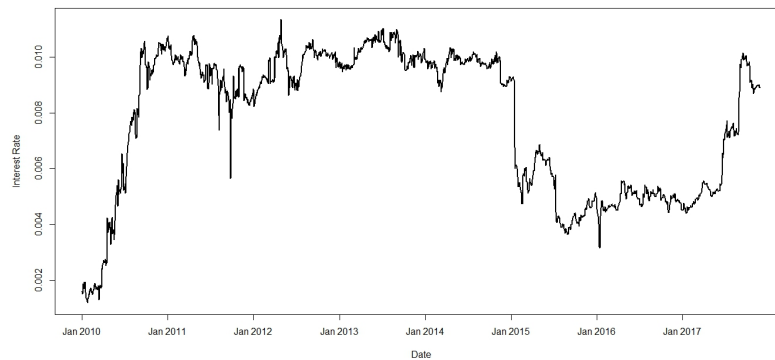


Figure 1: Plot of 3 month term structure

can be observed that the data has a low volatility as the standard deviation is very low with a small range of values between the maximum and minimum values.

Table 1: Descriptive statistics

Mean	Std Dev.	Min	Max	Range	Kurtosis	Skewness
0.007742	0.002565	0.001196	0.011334	0.010138	2.021842	-0.611756

4.2 Parameter Estimation

The parameter estimates of the Pearson-Sun and the CIR models are obtained via the method of maximum likelihood. From Tables 2 and 3, it can be observed that the parameter estimates of both the CIR and Pearson-Sun models are all significant at a 5% significance level. Most significant in this result is the fact that r_b is significantly different from zero. It can also be noted that the

standard errors are all less than 4%. According to [7], this is a good indication that the calibration procedure was successful.

Table 2: Estimates under Pearson-Sun model

Parameter	Estimate	Std Error	p-value
α	1.250465	0.536051	0.019662
μ	0.009884	0.000799	0.000000
r_b	0.001116	0.000785	0.016230
σ	0.046666	0.000744	0.000000

Table 3: Estimates under CIR model

Parameters	Estimate	Std Error	p-value
α	1.212284	0.013275	0.000000
μ	0.008963	0.001302	0.000000
σ	0.046676	0.000741	0.000000

Table 4 below also shows the descriptive statistics after simulating the data with the estimated parameters. It can be observed that the statistics of the simulated Pearson-Sun model are much closer to those of the real data. This is an indication that the Pearson-Sun model provides a better fit to the real data used.

Table 4: Descriptive statistics of real and simulated data

Data	Mean	Std Deviation	Min.	Max.	Range
Real Data	0.007742	0.002566	0.001196	0.011334	0.010138
PS	0.007963	0.002525	0.001396	0.012545	0.011149
CIR	0.007314	0.002215	0.001520	0.013003	0.011483

4.3 Model Comparison

The Pearson-Sun and CIR models only approximate the real interest rates, hence, the main goal of the above estimation procedure of the two competing models is to choose a model which best describes the characteristics of the data used.

From Table 5, it can be observed that the Pearson-Sun model has the least among the comparison measures. This means that the Pearson-Sun model minimises loss of the information contained in the observed data as compared

to the CIR model.

Table 5: Comparison measures

Model	AIC	BIC	RMSE	MAPE
PS	-27826.87	-27810.11	0.001225	14.397640
CIR	-27649.29	-27626.94	0.001394	17.221690

Figure 2 shows the plots of the real and simulated data sets. It can be observed that the plot from the simulated data under the Pearson-Sun model provides a better fit to the original data.

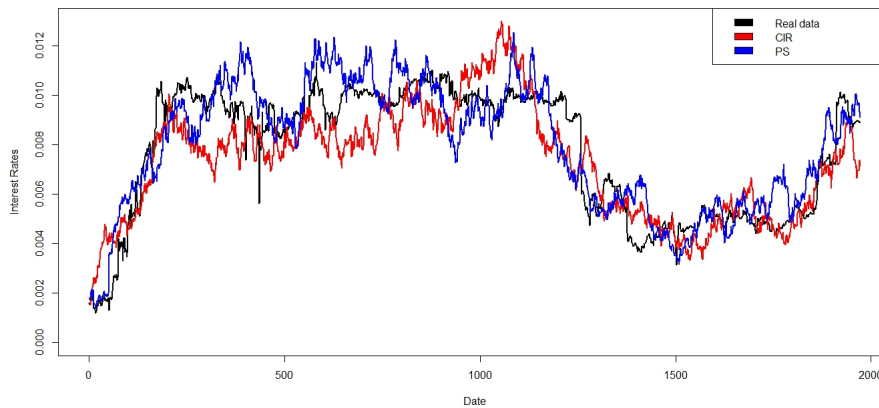


Figure 2: Plot of real and simulated data

4.4 Caps and Floor Prices

To simulate caps and floors prices, the following were considered. The time to maturity ranged from three months (0.25 years) to 10 years with an interval of 0.25 years (three months), with a tenor of 3 months. Again, a nominal value of one (1) was considered with a strike rate of 0.5%.

Figure 3(a) shows the model price of a cap. It shows a smooth increasing cap prices over the period. Since the mean of the data used is above the strike rate, an investor holding an interest rate cap with a rate of 0.5% will be in an advantageous position.

Also, Figure 3(b) shows the prices of floors over the period. There are lower prices for interest rate floors as compared to cap prices over the period.

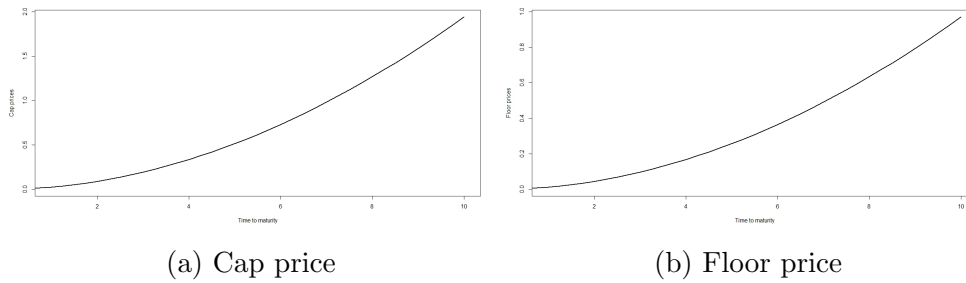


Figure 3: Model prices

As compared to interest rate caps, floors are sold at a lower price over the period. This can be observed from the maximum values shown in Table 6.

Table 6: Descriptive statistics of prices

Derivative	Mean	Std Deviation	Min	Max
Cap	0.691397	0.600494	0.002497	1.941187
Floor	0.345699	0.300247	0.001248	0.970593

5 Conclusion

In this paper, the price of interest rate caps and floors under the Pearson-Sun model were obtained. The Pearson-Sun model is an extension of the CIR model that preserves the CIR drift structure and introduces a lower bound to the model. The study shows that the Pearson-Sun model provides a much better description of the data than the CIR model. The Pearson-Sun model will therefore provide better pricing results in financial markets as compared to CIR model. The prices of the caps and floors were simulated based on the estimated parameters.

The study simulates the prices of caps and floors. However, these prices were not compared with the market cap and floor prices. The study therefore recommends the comparison of the model prices with market prices. Finally, it is recommended that a comparison of the pricing performance of the Pearson-Sun model with other well known interest rate models including models that allow for initial term structure such as the CIR++.

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