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A Reduced Form of the Three Factor Commodity Derivative Valuation Model

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Abstract

This study build a reduced form of three factor valuation model by explicitly taking into account the unobservable character of the convenience yield. The spot price process, the instantaneous convenience yield and CIR interest rate process are taken in the reduced form of three factor valuation model. The CIR interest rate process prevents interest rate from being negative. We simulate the reduced form three factor valuation model by using Milstein and Euler discretization schemes. We study the performance of Milstein and Euler discretization schemes theoretically and empirically in reduced form three factor valuation model. The Milstein discretization scheme has better approximation than Euler discretization scheme in reduced form three factor valuation model. As the time of maturity, T, is less and the time interval decreases the result obtained from the simulation of reduced form three factor valuation model for spot price process, convenience yield and interest rate process has better approximation.

AMS subject classification:
Keywords: Reduced Form Model, Discretization, Milstein scheme, Euler scheme.

1. Introduction

Assume we have (Ω, F, P) complete probability space and a finite time interval $[0, T]$. Assume again we have three stochastic processes i.e, the spot price process, S_t, instantaneous convenience yield process, δ_t and risk free interest rate, r_t. This study develops a reduced form three factor valuation model and find numerical solution of the joint stochastic differential equations above using the two known discretization schemes i.e, Euler and Milstein schemes. Reduced form models are trying to identify the relevant state variables or factors. A growing number of empirical studies return predictability pointed out the important role of the convenience yield. The spot price and the convenience yield are therefore the two commodity used state variables in pricing models. Current studies showed that there are a problem while finding analytical solution of a given stochastic differential equations(Anqi Shao,2012, Akinbo B.J, Faniran T and Ayoola E.O, 2015, Aurélien Alfonsi, 2005), especially for CIR model since the process has no closed form solution. Hence the researchers are forced to study the process numerically using discretization techniques in order to get best approximation solution for a give process.[1] There are many methods which can allow us to find numerical solution for continuous time processes but this study used only Euler and Milstein discretization schemes because of their convergence to the true result.

The Cox-Ingersoll-Ross(CIR) model is a diffusion process suitable for modeling the term structure of interest rate(Anqi Shao, 2012). The dynamics for CIR model is given by,[2]

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t \tag{1.1}$$

For $\kappa > 0$, $\theta > 0$, $\sigma > 0$ and Wiener process W. This process has some appealing properties from an practical point of view (Aurérien Alfonsi, 2005) i.e, the interest rate remains positive and the CIR process is mean reverting in nature (Cox et al. 1985). The condition $2\kappa\theta > \sigma^2$ would ensure that the origin is inaccessible to the process so that we can grant that the process X_t stays non-negative. One of our challenge, when we are simulating CIR model was explained by Anqi Shao, in his article. One drawback for simulation of CIR model is the process is not explicitly solvable (Aurérien Alfonsi, 2005). Due to this drawback we need to look further and proceed to find the method used to find numerical solution of the process to tackle this problem. The problem can be solved by using Diop A. and Deelstra and Delbaen's approach i.e, Diop A. approach

$$X_{t+1} = |X_t + \frac{T}{n}(a - \kappa X_t) + \sigma\sqrt{X_t}(W_{t+1} - W_t)| \tag{1.2}$$

Deelstra and Delbaen approach

$$X_{t+1} = X_t + \frac{T}{n}(a - \kappa X_t) + \sigma\sqrt{X_t > 0}(W_{t+1} - W_t) \tag{1.3}$$

[1] refer Alfonsi, 2005

[2] refer Cox et al. 1985

One of the simplest example of discretization is Euler scheme-Maryyama discretization (Akinbo B.J., Faniran T. and Ayoola E.O., 2015). We assume stochastic differential equation,

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{1.4}$$

Discretizing the above process by Euler scheme on $0 \leq t \leq T$ for a given discretization, $0 \leq t_1 < t_2 < \cdots < t_n \leq T$ of final time interval [0,T] is given as follows,

$$X_t = X_{t-1} + \mu X_{t-1}\Delta t + \sigma X_{t-1}\sqrt{\Delta t} n_{X,t-1} \tag{1.5}$$

Where,
Δ is the length of the time discretization subinterval and,
$n_{X,t-1}$ is a standard normal random variable.
Euler-Maruyama scheme is is a method used to approximate numerical solution of a continuous time processes. In practice, many stochastic differential equations are not explicitly solvable like CIR model therefore we can not get an analytical solution to a given continuous time process (Anqi Shao, 2012).

Like Euler-Maruyama discretization scheme Milstein scheme also used to find the numerical solution of a given stochastic differential equations. For the above stochastic differential equation (1.4) we have,

$$X_t = X_{t-1} + \mu X_{t-1}\Delta t + \sigma X_{t-1}\sqrt{\Delta t} n_{X,t-1} + \frac{1}{2}\sigma^2 X_{t-1}(\Delta t^2_{nX,t-1} - \Delta t) \tag{1.6}$$

Where, Δt and $n_{X,t-1}$ are as defined above. Milstein scheme has strong order of convergence one and Milstein will converges to the correct stochastic solution process faster than Euler-Maruyama as the step size Δt goes to zero (Akinbo B.J., Faniran T. and Ayoola E.O., 2015). Due to the property of it's strong convergency, Milstein discretization scheme give better approximation for a given continuous time processes.

Proposed reduced form three factor Model

In this study our contributions to the existing literature are three folds: First, we extend the Schwartz (1997) three factor model by adding new feature, which is Vasicek interest rate process in Schwartz (1997) model is replaced by mean reverting Cox-Ingersoll-Ross (CIR) process as described by Cox et al. (1985). This new feature prevents negative interest rate. Second, we provide numerical solution for reduced form three factor valuation model by using two known discretization, i.e, Euler-Maruyama and Milstein discretization techniques. Third, we study the strong convergence between Euler-Maruyama and Milstein discretization methods.

Financial Market

Assume we have (Ω, F, P) complete probability space with a standard filtration $F = \{\mathcal{F}(t) : t \in [0, T]\}$, a finite time period $[0, T]$. Assume we have three stochastic processes i.e, the spot price process of the underlying commodity, S, the instantaneous convenience yield process, δ, and the instantaneous interest rate process, r as presented in Schwartz (1997). First we discuss joint stochastic process for the two state variables i.e, the spot price process and the instantaneous convenience yield under the equivalent martingale measure can be expressed as[3]:

$$\frac{dS(t)}{S(t)} = (\mu - \delta(t))dt + \sigma_s dZ_s(t) \tag{1.7}$$

$$d\delta(t) = \kappa(\alpha - \delta(t))dt + \sigma_\delta dZ_\delta(t) \tag{1.8}$$

with initial conditions $S(0) \equiv S_0$ and $\delta(0) \equiv \delta_0$. Two correlated standard Brownian motions Z_s and Z_δ such that, $dZ_s dZ_\delta = \rho dt$, here ρ[4] stands for correlation coefficient between the two Brownian motions. $\kappa > 0$ is the magnitude of the speed of adjustment of the long run mean α, σ_s and σ_δ represents, respectively, constant, strictly positive, instantaneous standard deviation of the spot price and convenience yield.

Interest rates have an impact on spot commodity prices and on convenience yields.[5]

The reduced form three factor valuation model can be expressed as follows,

$$\frac{dS(t)}{S(t)} = (r - \delta(t))dt + \sigma_s dZ_s^*(t) \tag{1.9}$$

$$d\delta(t) = \kappa(\hat{\alpha} - \delta(t))dt + \sigma_\delta dZ_\delta^*(t) \tag{1.10}$$

$$dr(t) = a(m^* - r(t))dt + \sigma_r\sqrt{r(t)}dZ_r^*(t) \tag{1.11}$$

With initial conditions $S(0) \equiv S_0$, $\delta(0) \equiv \delta_0$ and $r(0) \equiv r_0$. Where, $\hat{\alpha} = \alpha - \frac{\lambda}{\kappa}$, three correlated standard Brownian motions, $dZ_s^* dZ_\delta^* = \rho_1 dt$, $dZ_\delta^* dz_r^* = \rho_2 dt$ and $dZ_s^* dZ_r^* = \rho_3 dt$. a is the speed of adjustment, m^* the risk adjusted mean short rate of the interest rate and σ_r is the constant, strictly positive, instantaneous standard deviation of interest rate, $r(t)$. The SDE of the short rate follows a mean-reverting process as Cox-Ingersoll-Ross(CIR).[6] If $2am^* > \sigma_r^2$, the CIR process is strictly positive, otherwise non-negative. Hence, the CIR interest rate model depicts the actual condition of the market where interest rate is non-negative unlike Vasicek interest rate model. The CIR model is mean reverting in nature. If the process deviates from the stationary mean level m^*, it is brought back to m^* at the rate of a.

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[3]see Schwartz (1997)

[4]Positive correlation between spot price and convenience yield is induced by the level of commodities: when inventories of the commodity decreases the spot price should increase since the commodity is scare and the convenience yield should also increase since futures prices will not increase as much as the spot price, and vice versa (Carmona and Ludkovski, 1991)

[5]Equilibrium models of commodity contingent claims assume that interest rates are zero or constant and do not study the relation between convenience yields and interest rates (Anh Ngoc Lai and Constantin Mellios,2015)

[6]see John C. Cox,Jonathan E. Ingersoll, Jr. and Stephen A. Ross, 1985

Euler and Milstein discretization schemes

Euler discretization scheme

Using an Euler discretization to simulate CIR process gives rise to the problem that while the process itself is guaranteed to be nonnegative, the discretization is not. General schemes, such as the Euler scheme or the Milstein scheme are in general not well defined because they can lead to negative values for which the square root is not defined (Anoi Shao, 2012). To tackle this problem in our simulation we use Diop's[7] approach which is 'reflection scheme' taking the norm of the discretization[8]. Therefore the straightforward Euler discretization scheme for valuation model is given by,

$$S_t = S_{t-1} + (\mu - \delta_{t-1})S_{t-1}\Delta t + \sigma_s S_{t-1}\sqrt{\Delta t}_{nS,t-1} \tag{1.12}$$

$$\delta_t = \delta_{t-1} + \kappa(\alpha - \delta_{t-1})\Delta t + \sigma_\delta \rho_1 \sqrt{\Delta t}_{nS,t-1} + \sigma_\delta \sqrt{1-\rho_1^2}\sqrt{\Delta t}_{n\delta,t-1} \tag{1.13}$$

$$r_t = r_{t-1} + a(m - r_{t-1})\Delta t + \sigma_r \rho_3 \sqrt{r_{t-1}}\sqrt{\Delta t}_{nS,t-1} + \sigma_r \sqrt{r_{t-1}}\sqrt{1-\rho_3^2}\sqrt{\Delta t}_{nr,t-1} \tag{1.14}$$

Where $n_{S,t-1}$, $n_{\delta,t-1}$ and $n_{r,t-1}$ are independent and identically distributed standard normal random variables.

Milstein discretization scheme

Milstein discretization[9] scheme is given by,

$$S_t = S_{t-1} + (\mu - \delta_{t-1})S_{t-1}\Delta t + \sigma_s S_{t-1}\sqrt{\Delta t}_{nS,t-1} + \frac{1}{2}\sigma_s^2 S_{t-1}(\Delta t^2_{nS.t-1} - \Delta t) \tag{1.15}$$

$$\delta_t = \delta_{t-1} + \kappa(\alpha - \delta_{t-1})\Delta t + \sigma_\delta \rho_1 \sqrt{\Delta t}_{nS,t-1} + \sigma_\delta \sqrt{1-\rho_1^2}\sqrt{\Delta t}_{n\delta,t-1} \tag{1.16}$$

$$r_t = r_{t-1} + a(m - r_{t-1})\Delta t + \sigma_r \rho_3 \sqrt{r_{t-1}}\sqrt{\Delta t}_{nS,t-1} + \sigma_r \sqrt{r_{t-1}}\sqrt{1-\rho_3^2}\sqrt{\Delta t}_{nr,t-1} \tag{1.17}$$

Where $n_{S,t-1}$, $n_{\delta,t-1}$ and $n_{r,t-1}$ are as stated above.

Simulation

To simulate reduced form three factor valuation model we use Euler and Milstein discretization representations listed above of the model with different discretization intervals. For both discretization techniques, we use the same time interval [0,1]. For each discretization schemes[10], we choose $\Delta t = 10^{-3}$ and use 10^3 simulation paths and $\Delta t = 10^{-1}$ and use 10^2 simulation paths.

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[7]refer Berkaoui A., Bossy M. and Diop A., 2008

[8]refer Aurélien Alfonsi, 2005

[9]as stated in Ola Elerian,1998

[10]The discretization alternatives with either the first-order Euler's approximation or the Milstein's approximation formats introduce discretization errors into the simulation and have higher computational cost because need small Δt.

Milstein discretization simulation for $T = 1$

We choose the following parameters to generate the trajectories in Milstein discretization scheme for valuation model.

Table 1: Parameters of valuation model for Milstein and Euler discretization scheme

σ_s	σ_δ	σ_r	ρ_1	ρ_2	ρ_3	κ	$\hat{\alpha}$	a	m^*
0.25	0.15	0.1	0.24	0.3	0.08	0.3	1	0.18	0.76

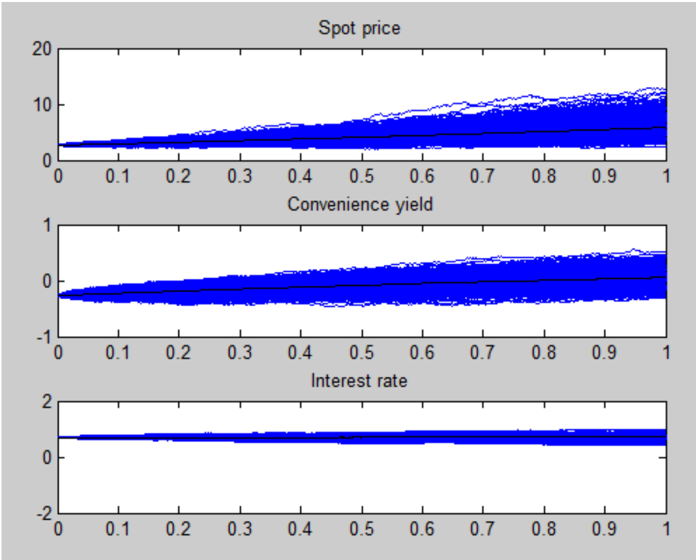

Figure 1: Milstein discretization simulation for valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

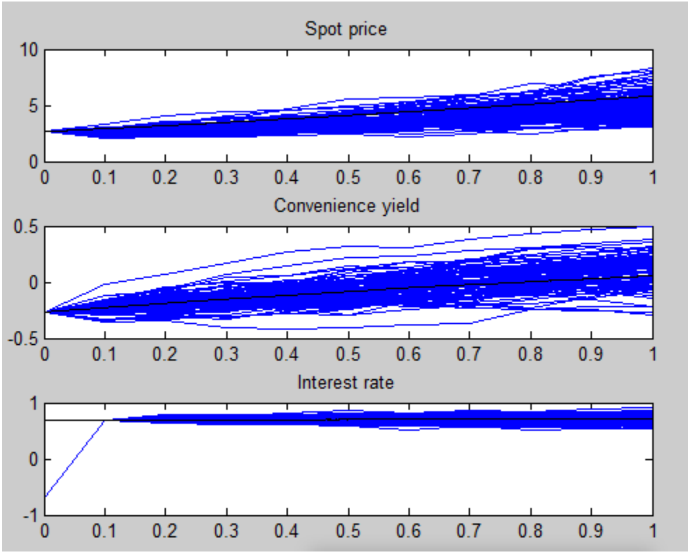

Figure 2: Milstein discretization simulation for valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

Euler discretization simulation for $T = 1$

We choose the same parameters as Milstein discretization scheme listed above.

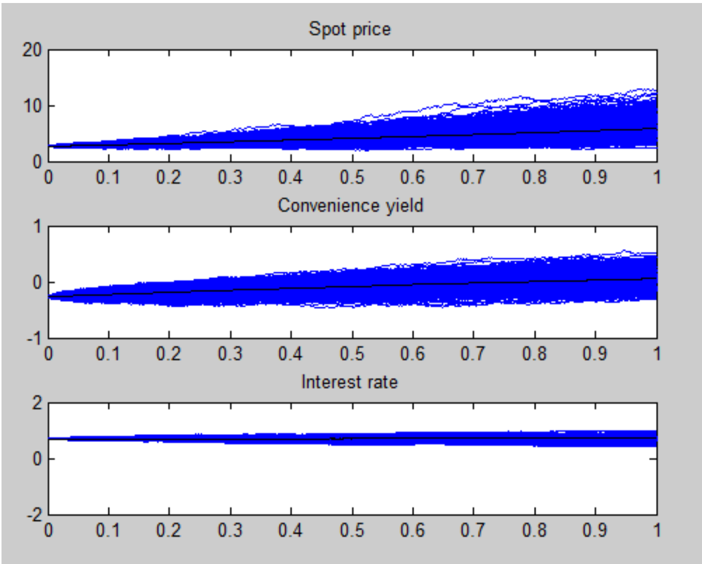

Figure 3: Euler discretization simulation for valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

For above figures (1), (2), (3) and (4) lines indicated by blue represents the simulation paths and the line indicated by black represents the true mean of the factor spot price, S.

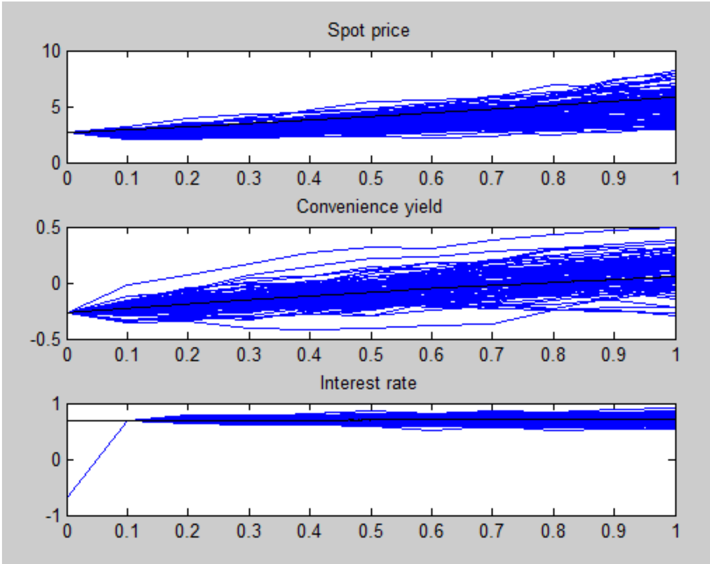

Figure 4: Euler discretization simulation for valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

Simulation results for both Milstein and Euler schemes as maturity expands Euler discretization simulation for $T = 5$

$T = 5$

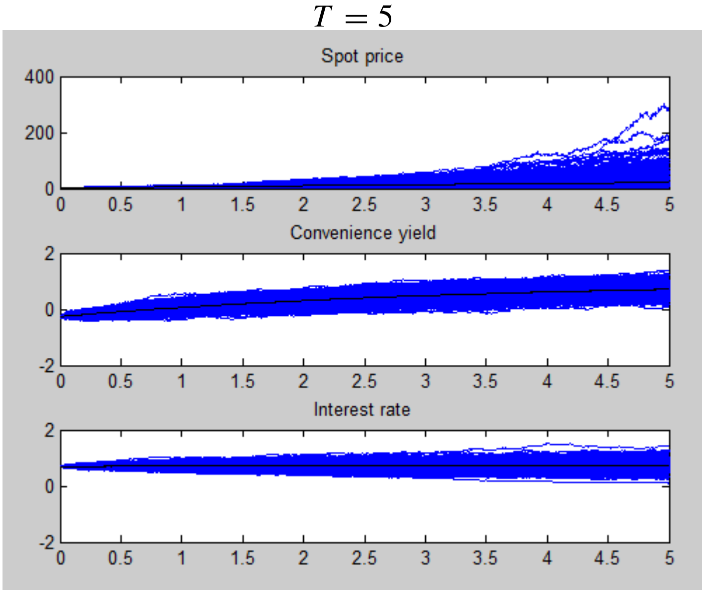

Figure 5: Euler discretization simulation for valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

$T = 5$

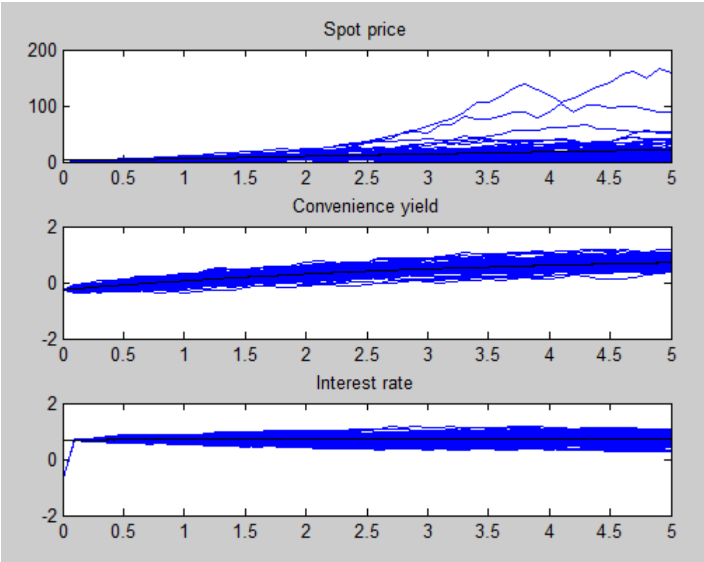

Figure 6: Euler discretization simulation for valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

Milstein discretization simulation for $T = 5$

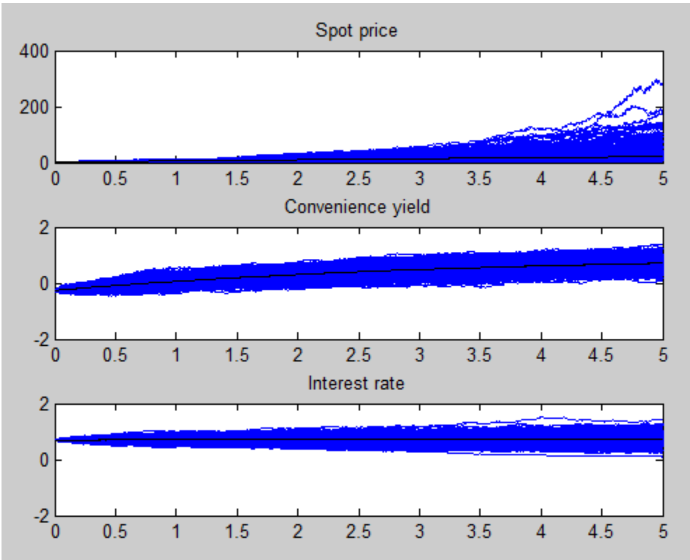

Figure 7: Milstein discretization simulation for valuation model with $\Delta t = 10^{-3}$ and 10^3 simulation paths

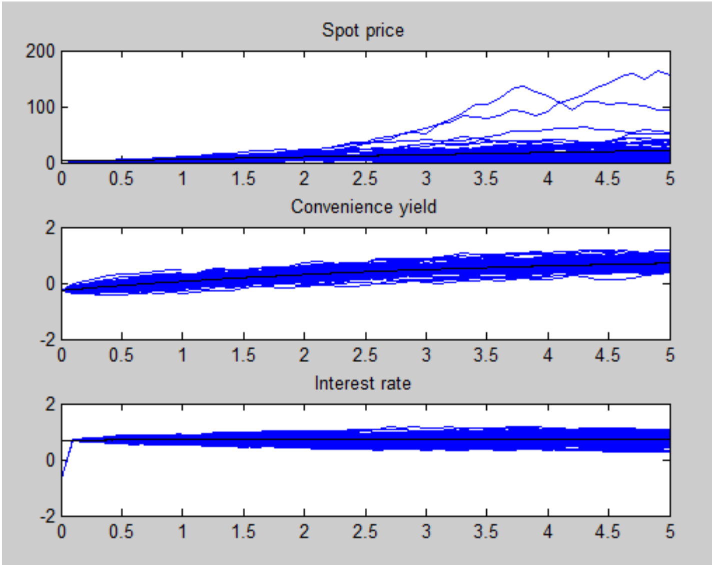

Figure 8: Milstein discretization simulation for valuation model with $\Delta t = 10^{-1}$ and 10^2 simulation paths

$T = 25$

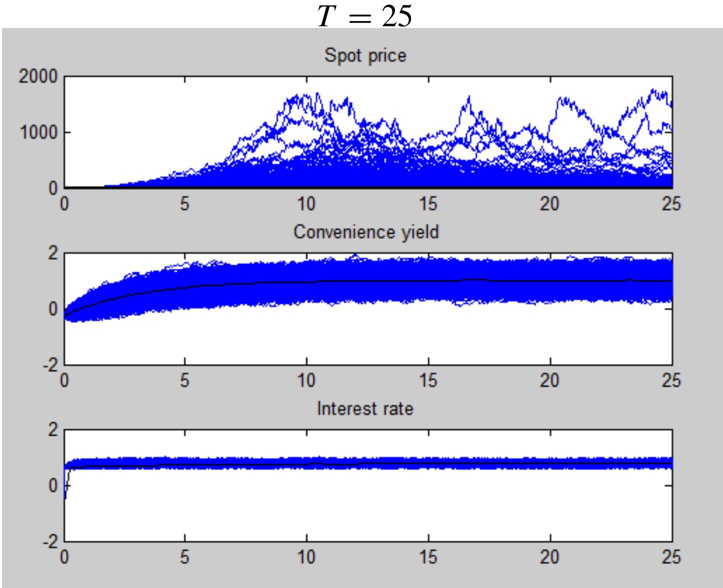

Figure 9: Simulation for reduced form three factor valuation model with $\Delta t = 0.025$ and 10^4 simulation paths

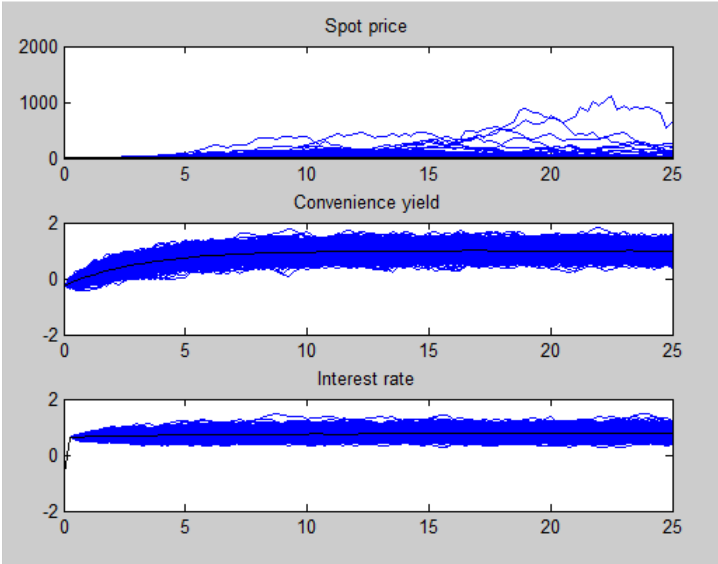

Figure 10: Simulation for reduced form three factor valuation model with $\Delta t = 0.25$ and 10^3 simulation paths

2. Simulation results

Table 2: Simulation results for the first ten simulation paths for $\Delta t = 10^{-5}$ and T=1

t	S	δ	r	True mean of S	True mean of δ	True mean of r	Ab. error in S	Ab. error in δ	Ab. error in r
0	2.658897758	−0.265611268	0.69674544	2.658897758	−0.265611268	0.69674544	0	0	0
1×10^{-5}	2.659807581	−0.265724802	0.696868195	2.658971699	−0.26563021	0.696763447	0.000835882	0.063735992530	0.000104748
2×10^{-5}	2.659391766	−0.265976129	0.697051102	2.659104144	−0.265618158	0.696735733	0.000287622	0.000357971	0.000315369
3×10^{-5}	2.659731848	−0.266181455	0.697104729	2.658997712	−0.265607887	0.696738081	0.000734135	0.000573568	0.000366649
4×10^{-5}	2.658860786	−0.266142717	0.697045966	2.658904728	−0.265602877	0.696747529	0.02960781932	0.00053984	0.000298438
5×10^{-5}	2.654995665	−0.266367026	0.697017383	2.658974969	−0.265628119	0.696749456	0.003979304	0.000738907	0.000267927
6×10^{-5}	2.652938128	−0.266872281	0.696629212	2.659107973	−0.265578497	0.696762256	0.006169845	0.001293784	0.000133044
7×10^{-5}	2.655396543	−0.266828903	0.697016206	2.659141004	−0.265584732	0.696760074	0.00374446	0.001244171	0.000256132
8×10^{-5}	2.653494379	−0.266616827	0.697019088	2.658894072	−0.265621916	0.696759831	0.005399692	0.000994912	0.000259256
9×10^{-5}	2.651565607	−0.266719849	0.697117397	2.658925909	−0.265581912	0.696783244	0.007360302	0.001137937	0.000334153

Table 3: Simulation results for the first ten simulation paths for $\Delta t = 0.1$ and $T = 25$

t	S	δ	r	True mean of S	True mean of δ	True mean of r	Ab. error in S	Ab. error in δ	Ab. error in r
0	2.658897758	−0.265611268	0.69674544	2.658897758	−0.265611268	−0.265611268	0	0	0.962356708
0.1	2.954985686	−0.202849649	0.659492936	2.91942026	−0.229401124	−0.229401124	0.035565426	0.026551475	0.88889406
0.2	2.975497172	−0.251518667	0.644811285	3.181260525	−0.196434354	−0.196434354	0.205763353	0.055084313	0.841245639
0.3	3.453187632	−0.144718969	0.646907219	3.480092229	−0.158283892	−0.158283892	0.026904597	0.013564923	0.80519111
0.4	3.624502244	−0.079915515	0.65186253	3.763128822	−0.124788895	−0.124788895	0.138626579	0.04487338	0.776651425
0.5	3.93889452	−0.031159645	0.632323953	4.079599242	−0.090808489	−0.090808489	0.140704722	0.059648844	0.723132442
0.6	4.505903711	0.021775318	0.611972641	4.4167477	−0.058013842	−0.058013842	0.089156011	0.07978916	0.669986483
0.7	5.051100521	0.071794115	0.590921649	4.80783989	−0.026132385	−0.026132385	0.243260631	0.0979265	0.617054034
0.8	5.512461959	0.10741752	0.608322118	5.183422184	0.007771102	0.007771102	0.329039775	0.099646418	0.600551016
0.9	6.370158275	0.159780054	0.614945852	5.568159396	0.039803189	0.039803189	0.801998879	0.119976865	0.575142663

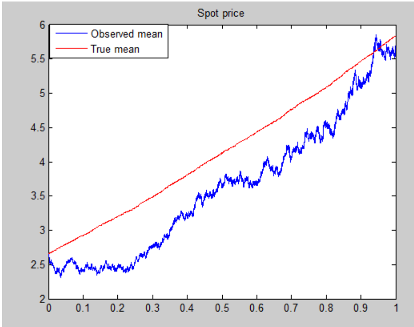

Figure 11: Simulation of spot price for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T = 1$

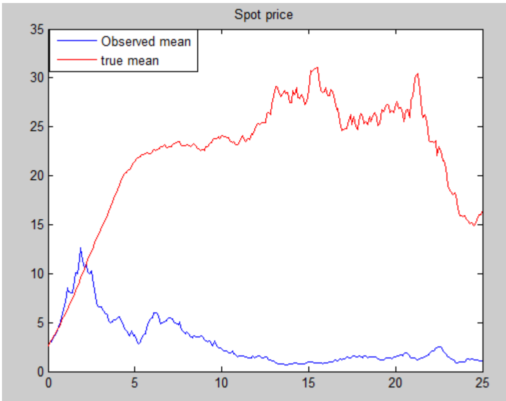

Figure 12: Simulation of spot price for $\Delta t = 0.1$ and 10^3 simulation paths, $T = 25$

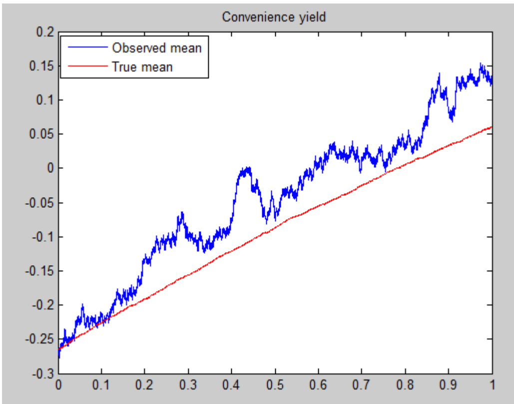

Figure 13: Simulation of convenience yield for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T = 1$

3. Conclusion

We can easily noted that the mean of 10^3 simulation paths indicated in blue is closer to the true mean indicated in black than the mean of 10^2 simulation paths for Milstein and

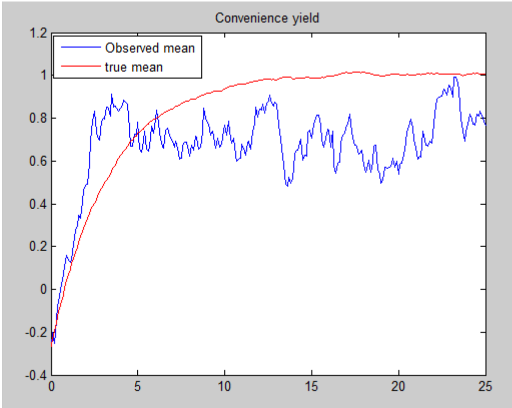

Figure 14: Simulation of convenience yield for $\Delta t = 0.1$ and 10^3 simulation paths, $T = 25$

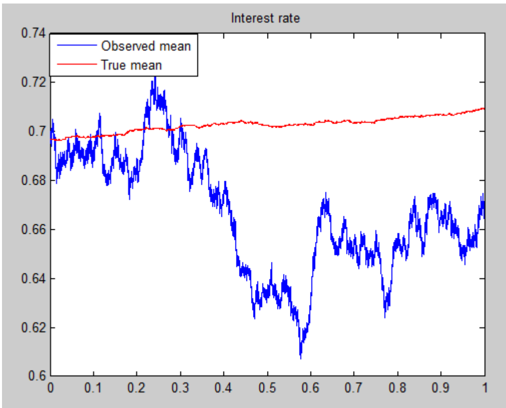

Figure 15: Simulation of interest rate for $\Delta t = 10^{-5}$ and 10^3 simulation paths, $T = 1$

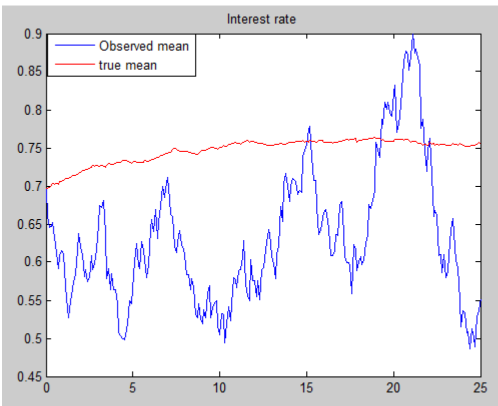

Figure 16: Simulation of interest rate for $\Delta t = 0.1$ and 10^3 simulation paths, $T = 25$

Euler discretization scheme. We can conclude that as the time interval (Δt) decreases the simulation result for both Milstein and Euler schemes achieves better approximations.

Since in reduced form three factor valuation model the diffusion coefficients in the spot price process, S, and interest rate process, r, unlike Schwartz (1997) three factor model, are not constant, the Milstein scheme and Euler scheme generates different results for spot price and interest rate so that we easily distinguish Milstein and Euler scheme discretization.

It can be concluded that as maturity, T, increases, there appear more uncertainty in both Euler and Milstein schemes simulation results. These leads to a less accuracy in the simulations obtained by both Milstein and Euler schemes. To get the best approximation for spot price, convenience yield and interest rate in reduced form three factor valuation model one can use less maturity and less time interval for both dscretization schemes.

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