Nonparametric Prediction Interval for Conditional Expected Shortfall Admitting a Location-Scale Model using Bootstrap Method

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ABSTRACT

In financial risk management, the expected shortfall is a popular risk measure which is often considered as an alternative to Value-at-Risk. It is defined as the conditional expected loss given that the loss is greater than a given Value-at-Risk (quantile). In this paper at hand, we have proposed a new method to compute nonparametric prediction bands for Conditional Expected Shortfall for returns that admits a location-scale model. Where the location (mean) function and scale (variance) function are smooth, the error term is unknown and assumed to be uncorrelated to the independent variable (lagged returns). The prediction bands yield a relatively small width, indicating good performance as depicted in the literature. Hence, the prediction bands are good especially when the returns are assumed to have a location-scale model.

Keywords: *Bootstrap, Expected Shortfall, Location-Scale Model, Nonparametric Prediction Intervals, Value-at-Risk*

Category: **Statistics and Data Science**

1 Introduction

Expected Shortfall (ES) is often used in portfolio risk measurement, risk capital allocation and performance attribution. Value-at-Risk (VaR) is defined as the conditional quantile of the portfolio loss distribution for a given horizon (it could be a day or a week) and for a given coverage rate (for instance 0.01 or 0.05), and the ES is simply defined as the expected loss beyond the VaR. Thus, VaR and ES measures are clear expressions about the left tail of the return distribution.

The concept of bootstrapping hangs on the idea that the probability distribution function of the data set available is unknown.

Therefore, the problem of constructing nonparametric prediction bands for Conditional Expected Shortfall (CES) where the returns are assumed to have a location-scale model with heteroscedasticity, and also distribution of the error term is assumed unknown using bootstrap method is of interest in this paper.

2 The Nonparametric Predictive Intervals (NPIs) for Conditional Expected Shortfall

Definition 1: $\alpha - mixing$ **(Strong mixing)**

Let $\mathcal{F}_k^l$ be the $\sigma - algebra$ of events generated by $\{Y_i, k \leq i \leq l\}$ for $l > k$. The $\alpha - mixing$ coefficient introduced by [4] is defined as

$$\alpha(k) = \sup_{\mathcal{A}\in\mathcal{F}_1^i, \mathcal{B}\in\mathcal{F}_{i+k}^\infty} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)|.$$

The series is said to be $\alpha - mixing$ if

$$\lim_{k\to\infty} \alpha(k) = 0.$$

The dependence described by the $\alpha - mixing$ is the weakest as it is implied by other types of mixing.

In this paper, we assumed that the sequence $\{Y_t\}$ satisfies a certain weak dependence condition, the concept of strong mixing coefficients by [4] as defined above. We further assumed that returns considered here, Y_t, admit a location-scale representation given as

$$Y_t = m(X_t) + \sqrt{h(X_t)}\epsilon_t \quad (1)$$

where m and $h > 0$ are nonparametric functions defined on the range of X_t , ϵ_t is independent of X_t, and ϵ_t is an independent and identically distributed (iid) innovation process with $\mathbb{E}(\epsilon_t) = 0$, $\mathbb{V}ar(\epsilon_t) = 1$ and the unknown distribution function F_ϵ.

From equation (1) we have

$$CVaR(X)_\tau := Q_{Y|X}(\tau|x) = m(X_t) + \sqrt{h(X_t)}q(\tau) \quad (2)$$

where $Q_{Y|X}(\tau|x)$ is the conditional $\tau-$quantile associated with $F(y|x)$ and $q(\tau) = F_\epsilon^{-1}(\tau)$ is the $\tau-$quantile associated with the error distribution F_ϵ. The estimator of (2) and its properties has been studied in [6],

and

$$CES(X)_\tau \equiv \mathbb{E}(Y_t/Y_t > Q_{Y|X}(\tau|x), X_t = x) = m(X_t) + \sqrt{h(X_t)}\mathbb{E}(\epsilon_t|\epsilon_t > q(\tau)) \quad (3)$$

Estimation of $m(X)$ and $h(X)$ in equations (2) and (3) was studied by [1] and [2]. For estimation of the error term, see our paper for more [7] details.

With estimators of the mean function $m(X)$, the variance function $h(X)$ and the unknown error distribution, the estimator for Conditional Value-at-Risk (CVaR), discussed in [6] is given as

$$\widehat{CVaR}(x)_\tau := \hat{Q}_{Y|X}(\tau|x) = \hat{m}(x) + \hat{h}^{1/2}(x)\hat{q}(\tau) \quad (4)$$

therefore, the estimator for Conditional Expected Shortfall is

$$\widehat{CES}(x)_\tau := \mathbb{E}(Y_t/Y_t > \hat{Q}_{Y|X}(\tau|x), X_t = x) = \hat{m}(x) + \hat{h}^{1/2}(x)\mathbb{E}(\epsilon_t|\epsilon_t > \hat{q}(\tau)) \quad (5)$$

The mean and variance of the estimator (5) as discussed in [5] are as follows:

$$\mathbb{E}\Big[\widehat{CES}(x)_\tau\Big] \approxeq m(x) + \mathbb{E}(\epsilon_t|\epsilon_t > q(\tau))h(x) + \underbrace{\frac{b^2}{2}\mu_2(k)\big[m''(x) + \mathbb{E}(\epsilon_t|\epsilon_t > q(\tau))h''(x)^2\big]}_{=\text{Bias}} \quad (6)$$

So that,

$$Bias\Big(\widehat{CES}(x)_\tau\Big) \approxeq \frac{b^2}{2}\mu_2(k)\big[m''(x) + \mathbb{E}(\epsilon_t|\epsilon_t > q(\tau))h''(x)^2\big] \quad = B(x)_\tau \quad (7)$$

and

$$Var\Big(\widehat{CES}(x)_\tau\Big) \approxeq \frac{R(k)}{nbf(x)}\Big[\sigma^2(x) + \mathbb{E}(\epsilon_t|\epsilon_t > q(\tau))^2 h^2(x)\lambda^2(x)\Big] \quad = Var(x)_\tau \quad (8)$$

$\implies \widehat{CES}(x)_\tau \xrightarrow{d} CES(X)_\tau$, and by central limit theorem we have:

$$\sqrt{nb}\Big[\widehat{CES}(x)_\tau - CES(X)_\tau - B(x)_\tau\Big] \xrightarrow{d} \mathcal{N}\Big(0, Var(x)_\tau\Big) \quad (9)$$

2.1 Pivotal quantity (Pivot)

Def: Let $X = (X_1, ..., X_n)$ be random variables with unknown joint distribution F, and let $\theta(F)$ denote a real-valued parameter. A random variable $Q(X, \theta(F))$ is a pivot if the distribution of $Q(X, \theta(F))$ is independent of all parameters. That is, $X \sim F(x|\theta(F))$, then $Q(X, \theta(F))$ has the same distribution $\forall_{\theta(F)}$, see [8].

Consider the function estimator $\widehat{CES}(x)_\tau$ in (5), the asymptotic distribution of a pivotal quantity are used to construct confidence intervals (CIs). Let us defined $Q\Big(CES(X)_\tau, \widehat{CES}(x)_\tau\Big)$ to be the pivotal statistic given as

$$Q\Big(CES(X)_\tau, \widehat{CES}(x)_\tau\Big) = \frac{\widehat{CES}(x)_\tau - CES(X)_\tau}{\sqrt{Var(x)_\tau}} \tag{10}$$

where $Var(x)_\tau$ is the variance of the function estimator defined in (8).

2.2 Bootstrap Method

This strategy consist of estimating the distribution of the pivotal quantity given below

$$Q\Big(CES(X)_\tau, \widehat{CES}(x)_\tau\Big) = \frac{\widehat{CES}(x)_\tau - CES(X)_\tau}{\sqrt{\sigma^2(x) + \widehat{Var}(x)_\tau}} \tag{11}$$

using the bootstrap method. The distribution of (8) was approximated by the distribution of the bootstrapped statistics

$$\mathcal{T}\Big(\widehat{CES}(x)_\tau, \widehat{CES}^*(x)\Big) = \frac{\widehat{CES}^*(x) - \widehat{CES}(x)_\tau}{\sqrt{\sigma^2(x) + \widehat{Var}^*(x)_\tau}} \tag{12}$$

where $*$ denotes the bootstrap counterparts of the the estimates. Hence, we have the following Nonparametric Prediction Intervals with $(1 - \tau)$ asymptotic coverage probability.

$$\Big[\widehat{CES}(x)_\tau - \sqrt{\sigma^2(x) + \widehat{Var}^*(x)_\tau}\hat{q}(a),\ \ \widehat{CES}(x)_\tau + \sqrt{\sigma^2(x) + \widehat{Var}^*(x)_\tau}\hat{q}(b)\Big] \tag{13}$$

The Algorithm (Bootstrap)

1. Generate n data sets from the unknown probability model of the data generation process in (14), with independently identically distributed random errors form some

unknown probability distribution function (pdf) F_ϵ.

2. Calculate $\hat{m}(x)$ and $\hat{h}(x)$.

3. Generate the sequence of Standardized Nonparametric Residuals (SNR) $\{\hat{\epsilon}_t\}_{t=1}^n$, where

$$\hat{\epsilon}_t = \begin{cases} \dfrac{Y_t - \hat{m}(x)}{\sqrt{\hat{h}}(x)}, & \text{if } \hat{h}(x) > 0 \\ 0, & \text{if } \hat{h}(x) \leq 0 \end{cases}$$

and hence compute $\hat{q}(\tau) = \hat{F}_\epsilon^{-1}(\tau)$ and $\mathbb{E}(\epsilon_t|\epsilon_t > \hat{q}(\tau))$

4. Compute for each process the $\widehat{CES}^*(x)_j$, $j = 1, 2, \ldots, m$

5. Compute the average function, $\overline{\widehat{CES}}^*_m(x)$ given by:

$$\overline{\widehat{CES}}^*_m(x) = \frac{\widehat{CES}_1(x) + \widehat{CES}_2(x) + \cdots + \widehat{CES}_m(x)}{m}$$

6. and the standard error between the curves is:

$$SE = \sqrt{\frac{1}{m \times n} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\widehat{CES}^*(x_i)_j - \overline{\widehat{CES}}^*_m(x_i) \right)^2}$$

7. The lower and upper bounds of the NPIs at level τ are therefore given by

$$LB = \overline{\widehat{CES}}^*_m(x) - Q * \frac{SE}{\sqrt{m}}$$
$$UB = \overline{\widehat{CES}}^*_m(x) + Q * \frac{SE}{\sqrt{m}}$$

where $Q = z_{1-\frac{\tau}{2}}$ is the $(1 - \frac{\tau}{2})^{th}$ quantile for the standard normal distribution. For instance, $z_{1-\frac{\tau}{2}} = 1.96$ for $\tau = 0.05$.

3 Simulation Study

To examine the performance of our estimators, we conducted a simulation study considering the following data generating location-scale model

$$X_t = m(X_{t-1}) + h(t)^{1/2}\epsilon_t, \quad t = 1, 2, ..., n \tag{14}$$

where

$$m(X_{t-1}) = sin(0.5X_{t-1}), \quad \epsilon_t \sim t(\nu = 3), h(t) = h_i(X_{t-1}) + \theta h(X_{t-1}), \quad i = 1, 2$$

and $h_1(X_{t-1}) = 1 + 0.01X_{t-1}^2 + 0.5sin(X_{t-1})$, $h_1(X_{t-1}) = 1 - 0.9exp\big(-2X_{t-1}^2\big)$

X_t and $h(t)$ are set to zero (0) initially, then X_t is generated recursively from (10) above. The data generating process was used by [3] and also used by [7].

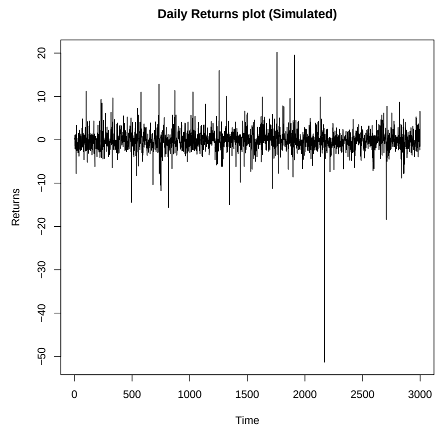

Figure 1: Plot of the simulated daily returns showing its evolution.

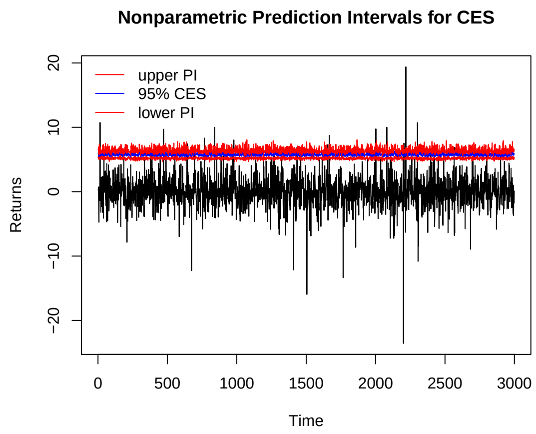

Figure 2: Graph showing the 95% Conditional Expected Shortfall in blue color, while the Upper and Lower Prediction Intervals in red color.

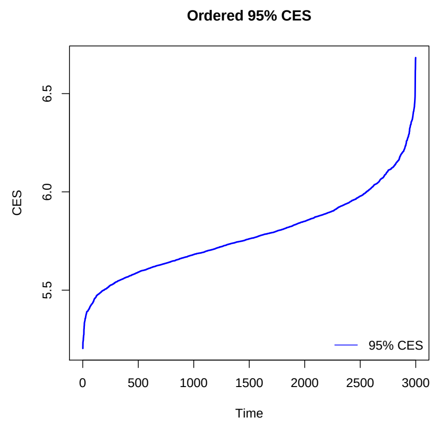

Figure 3: Graph showing the 95% ordered Conditional Expected Shortfall.

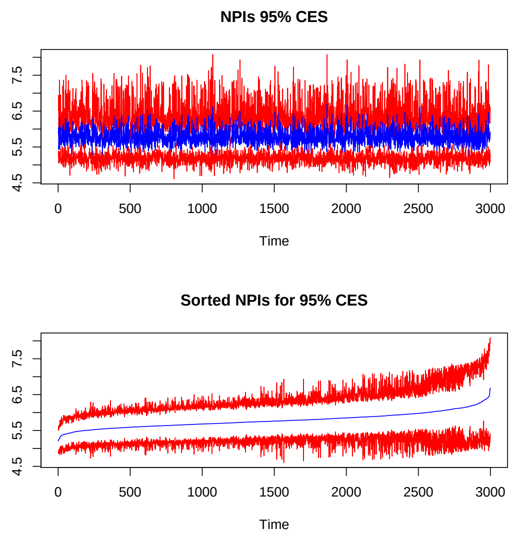

Figure 4: The red lines in the first and second panel shows the upper and lower prediction intervals, and the blue lines in both cases shows the 95% CES.

The time series plot of the simulated daily returns generated from the data generating process (14) is presented in **Figure 1**. Looking at **Figure 2** and **Figure 4**, the Nonparametric Prediction Intervals by bootstrap method for Conditional Expected Shortfall performs well. Clearly, the 95% CES is contained within the prediction bands. Plotting the ordered 95% CES in **Figure 3**, it shows the distribution of 95% CES over time. We can see that the width of the bands is considerably small affirming its good performance, conforming with what is obtainable in the literature on prediction intervals.

4 Conclusion

We proposed Nonparametric Prediction Bands for a conditional Expected Shortfall using bootstrap method. Our approach is based on returns on assets or portfolio that have a location-scale model. Simulation study was conducted and the prediction bands was found to perform very good.

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Disclosure statement

The authors declare that there is no conflict of interest regarding the publication of this paper

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