REFINED NODE POLYNOMIALS VIA LONG EDGE GRAPHS

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Abstract. The generating functions of the Severi degrees for sufficiently ample line bundles on algebraic surfaces are multiplicative in the topological invariants of the surface and the line bundle. Recently new proofs of this fact were given for toric surfaces by Block, Colley, Kennedy and Liu, Osserman, using tropical geometry and in particular the combinatorial tool of long-edged graphs. In the first part of this paper these results are for $\mathbb{P}^2$ and rational ruled surfaces generalised to refined Severi degrees. In the second part of the paper we give a number of mostly conjectural generalisations of this result to singular surfaces, and curves with prescribed multiple points.

1. Introduction

The Severi degree $n^{d,\delta}$ is the number of δ-nodal degree d curves in the projective plane $\mathbb{P}^2$ through $d(d+3)/2-\delta$ general points. More generally for a pair (S,L) of a complex projective surface a line bundle on S, the Severi degree $n^{(S,L),\delta}$ counts the number of δ-nodal curves in the linear system $|L|$ passing through $\dim|L|-\delta$ general points. In [DFI] is was conjectured that there are polynomials $n_\delta(d)$ in d, called *node polynomials*, such that $n^{d,\delta}=n_\delta(d)$, for d sufficiently large with respect to δ. In [Göt] it was conjectured that there are universal polynomials $t_\delta(x,y,z,w)$, such that for L sufficiently ample with respect to δ, $n^{(S,L),\delta}$ is obtained by substituting the intersection numbers L^2, LK_S, K_S^2, $\chi(\mathcal{O}_S)$: writing $n_\delta(S,L):=t_\delta(L^2,LK_S,K_S^2,\chi(\mathcal{O}_S))$ we should have $n_\delta(S,L)=n^{(S,L),\delta}$. The conjectures of [Göt] furthermore express the generating functions

$$n(d):=\sum_{\delta\geq 0}n_\delta(d)t^\delta,\qquad n(S,L):=\sum_{\delta\geq 0}n_\delta(S,L)t^\delta$$

in terms of some universal power series. $n(S,L)$ is multiplicative in the parameters, i.e.

$$n(S,L)=A_1(t)^{L^2}A_2(t)^{LK_S}A_3(t)^{K_S^2}A_4(t)^{\chi(\mathcal{O}_S)},\tag{1.1}$$

for some power series $A_i(t)\in\mathbb{Q}[[t]]$, and thus in particular

$$n(d)=A_1(t)^{d^2}A_2(t)^{-3d}A_3(t)^9A_4(t).\tag{1.2}$$

Furthermore explicit formulas for $A_1(t)$ and $A_4(t)$ are given in terms of modular forms.

We will call (1.1) and (1.2) the *multiplicativity* of $n(S,L)$ and $n(d)$. The Severi degrees of $\mathbb{P}^2$ and toric surfaces can be computed via tropical geometry, by the Mikhalkin correspondence theorem [Mik]. This was used in [FM] to prove the existence of the node polynomials $n_\delta(d)$, using Floor diagrams which are combinatorial devices for encoding

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tropical curves. The conjecture of [Göt] was proven in [Tze], [KST], using the methods of complex geometry. In [BCK] and [L] the Severi degrees $n^{d,\delta}$ were studied using long edge graphs, a modification of floor diagrams, giving an alternative proof for the multiplicativity of the generating function $n(d)$. This is done by taking the formal logarithm.

$$Q(d) := \log(n(d)) = \sum_{\delta\ge 1} Q_\delta(d)t^\delta.$$

The multiplicativity for $n(d)$ is equivalent to the statement that $Q_\delta(d)$ is a polynomial of degree 2 in d for all δ. This is proven in [BCK] and [L], giving the first purely combinatorial proofs of (1.2). In [LO] this result is generalized to a large class of toric surfaces, and a generalisation is given to toric surfaces with rational singularities. This note tries to extend these results to the refined Severi degrees defined in [GS] and [BG] and thus also to the Welschinger numbers.

The Welschinger numbers $W^{d,\delta}$ count δ-nodal degree d real curves in $\mathbb{P}^2$ through $d(d+3)/2-\delta$ real points with suitable signs, and $W^{(S,L),\delta}$ counts real δ-nodal curves in the linear system $|L|$ on a real algebraic surface S through a configuration of $\dim|L|-\delta$ real points. They are closely related to to the Welschinger invariants, deformation invariants defined in genus 0. The Welschinger numbers depend in general on the point configuration, but in [Mik] it is shown that, for a so called *subtropical* configuration of points, they coincide with the tropical Welschinger invariants $W^{\mathrm{trop}}_{d,\delta}$, $W^{\mathrm{trop}}_{(S,L),\delta}$, defined via tropical geometry (and these are independent of the tropical configuration of points). In future we will assume that we are dealing with a subtropical configuation of points .

In [GS] and [BG] refined Severi degrees $N^{d,\delta}(y)$, and $N^{(S,L),\delta}(y)$ for toric surfaces are introduced via tropical geometry. These are symmetric Laurent polynomials in a variable y, interpolating between the Severi degrees and the Welschinger numbers, i.e. $N^{(S,L),\delta}(1)=n^{(S,L),\delta}$, $N^{(S,L),\delta}(-1)=W^{(S,L),\delta}$. In [GS] analogues of the conjectures of [Göt] are formulated for the refined Severi degrees. In particular for $\delta\le 2d-2$ the $N^{d,\delta}(y)$ should be given by refined node polynomials $N_\delta(d;y)\in\mathbb{Q}[d,y^{\pm 1}]$. Similarly for pairs (S,L) of a smooth toric surface and a δ-very ample toric line bundle the conjectures say $N^{(S,L),\delta}(y)=N_\delta((S,L);y)$, for some polynomial in $N_\delta((S,L);y)$ in $L^2, LK_S, K_S^2, \chi(\mathcal{O}_S)$. In the case of $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ or a Hirzebruch surface Σ_m, these conjectures are (with weaker bounds) proven in [BG, Thm. 4.2].

We introduce generating functions for the refined node polynomials. Let

$$N(d)(y,t) := \sum_{\delta\ge 0} N_\delta(d;y)t^\delta, \qquad N(S,L)(y,t) := \sum_{\delta\ge 0} N_\delta(S,L;y)t^\delta.$$

In [GS] it is again conjectured that $N(S,L)(y,t)$ is multiplicative.

Conjecture 1. [GS] *There exist power series* $A_i(y,t) \in \mathbb{Q}[y^{\pm 1}][[t]]$, $i = 1,2,3,4$, *such that for all pairs* (S,L) *of a smooth toric surface and a toric line bundle we have*

$$N(S,L)(y,t) = A_1(y,t)^{L^2} A_2(y,t)^{LK_S} A_3(t)^{K_S^2} A_4(t)^{\chi(\mathcal{O}_S)},$$
$$N(d)(y,t) = A_1(y,t)^{d^2} A_2(y,t)^{-3d} A_3(y,t)^9 A_4(y,t). \tag{1.3}$$

Two of these power series are expressed in terms of Jacobi forms. One can rewrite (1.3) in a different way:

$$N_\delta(S,L;y) = \operatorname*{Coeff}_{q^{L(L-K_S)/2}} \left[DG_2(y,q)^{L(L-K_S)/2-\delta} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} B_3(y,q)^{\chi(\mathcal{O}_S)} \right] \tag{1.4}$$

Here $DG_2(y,q), B_i(y,q) \in \mathbb{Q}[y^{\pm 1}][[q]]$, and $DG_2(y,q)$, $B_3(y,q)$ are related to theta functions. For more details see Section 4.

In the first part of the current note we adapt the method of long edge graphs and the proofs of [BCK], [L], [LO] to refined Severi degrees, to prove the multiplicativity also for the $N(d)(y,t)$ and a weaker version of multiplicativity for rational ruled surfaces (see Theorem 24). We combine this with computer calculations of the refined Severi degrees and the Welschinger numbers of $\mathbb{P}^2$ and rational ruled surfaces. This allows to determine the refined node polynomials of $\mathbb{P}^2$ and rational ruled surfaces for low values δ, confirming the predictions of [GS] (see Corollary 31), and extending the results of [BG].

We then extend the results and conjectures to surfaces with singularities and to curves passing through (smooth or singular) points of S with higher multiplicity. This in particular includes a conjectural generalisation of the results of [LO] to the refined invariants. The conjectural formulas generalize (1.4). For every condition c that we can impose on the curves at a point of S, we get a power series $D_c(y,q) \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that the refined count of curves in $|L|$ on S satisfying conditions $c_1, \ldots, c_s$ will be given by

$$\operatorname*{Coeff}_{q^{L(L-K_S)/2}} \left[B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} B_3(y,q)^{\chi(\mathcal{O}_S)} \prod_{i=1}^{s} D_{c_i}(y,q) \right] \tag{1.5}$$

The formula (1.4) is the case that the conditions imposed are to pass through $L(L-K_S)/2-\delta$ general points, in particular $DG_2(y,q)$ is the power series corresponding to the condition of passing through a point.

2. Refined Severi degrees and long edge graphs

2.1. **Refined Severi degrees and Floor diagrams.** In [GS], [BG] refined Severi degrees were introduced. We will briefly recall some of the results and definitions.

A lattice polygon $\Delta \subset \mathbb{R}^2$ is a polygon with vertices of integer coordiates. The lattice length of an edge e of Δ is $\#(e \cap \mathbb{Z}^2) - 1$. We denote by $int(\Delta)$, $\partial(\Delta)$ its interior and its boundary. To a convex lattice polygon Δ one can associate a pair $S(\Delta)$, $L(\Delta)$ of a toric surface and a toric line bundle on $S(\Delta)$. The toric surface is defined by the fan given by the outer normal vectors of Δ. We have $\dim H^0(S(\Delta), L(\Delta)) = \#(\Delta \cap \mathbb{Z}^2)$. The arithmetic genus of a curve in $|L(\Delta)|$ is $g(\Delta) = \#(int(\Delta) \cap \mathbb{Z}^2)$. In [BG, Def. 3.8] refined Severi degrees $N^{\Delta,\delta}(y)$ are defined for any convex lattice polygon Δ. They are a count of tropical curves in $\mathbb{R}^2$ satisfying suitable point conditions with multiplicities which are Laurent polynomials in y. We also write $N^{S(\Delta),L(\Delta),\delta}(y) := N^{\Delta,\delta}(y)$. The $N^{\Delta,\delta}(y)$ interpolate between the Severi degrees (at $y=1$) and the tropical Welschinger numbers (at $y=-1$).

Example 2. In the following we will be concerned only with the following lattice polygons $\Delta_{c,m,d} = \{(x,y) \in (\mathbb{R}_{\geq 0} \mid y \leq d;\ x+my \leq md+c\}$, for $d, \geq 0, m \geq 0, c \geq 0$. These are so called h-transversal lattice polygons, i.e. all the slopes of the outer normal vectors of Δ are integers or $\pm\infty$. This covers three different cases:

(1) $d \geq 0$, $m=1$, $c=0$. In this case $S(\Delta_{0,1,d}) = \mathbb{P}^2$, $L(\Delta_{0,1,d}) = dH$, with H the hyperplane bundle on $\mathbb{P}^2$.
(2) $d \geq 0$, $m \geq 1$, $c=0$. In this case $S(\Delta_{0,m,d}) = \mathbb{P}(1,1,m)$, $L(\Delta_{0,m,d}) = dH$, with H the hyperplane bundle on $\mathbb{P}(1,1,m)$ with self intersection m.
(3) $d \geq 0, \geq 0, m \geq 0, c \geq 0$. In this case $S(\Delta_{c,m,d})$ is the rational ruled surface Σ_m. Let E be the class of a section with self intersection $-m$ and F the class of a fibre. Let $H := E+mF$. Then $L(\Delta) = cF + dH$.

Note that in some cases the same lattice polygon corresponds to different pairs of a surface and a line bundle, but by the above the refined Severi degree only depends on Δ.

In [BG] it was also shown that the refined Severi degrees can for h-transversal lattice polygons be computed in terms of Floor diagrams. Here we will not recall the definition of the refined Severi degrees as a count of tropical curves, but directly review them in terms of Floor diagrams which are very closely related to long-edge graphs. We will also restrict our attention to the lattice polygons $\Delta_{c,m,d}$ of Example 2, and thus to $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ and Σ_m. In the following we fix d, m, c and write $\Delta = \Delta_{c,m,d}$.

Definition 3. A Δ-*floor diagram* $\mathcal{D}$ consists of:

(1) A graph on a vertex set $\{1,\ldots,d\}$, possibly with multiple edges, with edges directed $i \to j$ if $i<j$. Edges e carry a weight $w(e) \in \mathbb{Z}_{>0}$.
(2) A sequence $(s_1,\ldots,s_d)$ of non-negative integers such that $s_1+\cdots+s_d = c$.
(3) (Divergence Condition) For each vertex j of $\mathcal{D}$, we have

$$\operatorname{div}(j) \overset{\text{def}}{=} \sum_{\substack{\text{edges } e\\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e\\ i \xrightarrow{e} j}} w(e) \leq m + s_j.$$

Notation 4. For an integer n we introduce the quantum number $[n]_y$ by

$$[n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}} = y^{n-1/2} + y^{n-3/2} + \ldots + y^{-n+3/2} + y^{-n+1/2}.$$

Definition 5. We define the *refined multiplicity* $\operatorname{mult}(\mathcal{D}, y)$ of a floor diagram $\mathcal{D}$ as

$$\operatorname{mult}(\mathcal{D}, y) = \prod_{\text{edges } e} \left([w(e)]_y\right)^2.$$

By definition $\operatorname{mult}(\mathcal{D}, y)$ is a Laurent polynomial in y with positive integral coefficients.

Definition 6. A *marking* of a floor diagram $\mathcal{D}$ is defined by the following four step process

Step 1: For each vertex j of $\mathcal{D}$ create s_j new indistinguishable vertices and connect them to j with new edges directed towards j.

Step 2: For each vertex j of $\mathcal{D}$ create $m + s_j - div(j)$ new indistinguishable vertices and connect them to j with new edges directed away from j. This makes the divergence of vertex j equal to m.

Step 3: Subdivide each edge of the original floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Denote the resulting graph $\widetilde{\mathcal{D}}$.

Step 4: Linearly order the vertices of $\widetilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram $\mathcal{D}$ such that, as before, each edge is directed from a smaller vertex to a larger vertex.

The extended graph $\widetilde{\mathcal{D}}$ together with the linear order on its vertices is called a marked floor diagram or *marking* of the floor diagram $\mathcal{D}$.

The *cogenus* of a marked floor diagram $\widetilde{\mathcal{D}}$ is $\delta(\widetilde{\mathcal{D}}) := \#(\Delta \cap \mathbb{Z}^2) - 1 - k$, where k is the total number of vertices of $\widetilde{\mathcal{D}}$ (this coincides with the cogenus of the tropical curve corresponding to $\widetilde{\mathcal{D}}$, see e.g. [BG2, Def. 4.2]). We count marked floor diagrams up to equivalence. Two markings $\widetilde{\mathcal{D}}_1$, $\widetilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists an automorphism of weighted graphs which preserves the vertices of $\mathcal{D}$ and maps $\widetilde{\mathcal{D}}_1$ to $\widetilde{\mathcal{D}}_2$. We denote $\nu(\mathcal{D})$ the number of markings $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ up to equivalence. Denote by $\mathbf{FD}(\Delta, \delta)$ the set of Δ-floor diagrams $\mathcal{D}$ with cogenus δ.

Theorem 7. *([BG, Thm. 5.7]) For $\Delta = \Delta_{c,m,d}$ as in Example 2 and $\delta \geq 0$, we have*

$$N^{\Delta,\delta}(y) = \sum_{\mathcal{D} \in \mathbf{FD}(\Delta,\delta)} \mathit{mult}(\mathcal{D}; y) \cdot \nu(\mathcal{D}).$$

2.2. **Caporaso-Harris type recursion.** In [BG] also a Caporaso-Harris type recursion is proven for the refined Severi degrees of $\mathbb{P}^2$, $\mathbb{P}(1, 1, m)$ and Σ_m, thus showing that they coincide with the refined Severi degrees as defined in [GS]. This recursion can be easily programmed in Maple, and has been extensively used in the course of this paper to find conjectural generating functions for the refined Severi degrees. In this section let S be $\mathbb{P}^2$, $\mathbb{P}(1, 1, m)$ and Σ_m. We first recall the notations.

By a *sequence* we mean a collection $\alpha = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers, almost all of which are zero. For two sequences α, β we define $|\alpha| = \sum_i \alpha_i$, $I\alpha = \sum_i i\alpha_i$,

$\alpha+\beta=(\alpha_1+\beta_1,\alpha_2+\beta_2,\ldots)$, and $\binom{\alpha}{\beta}=\prod_i\binom{\alpha_i}{\beta_i}$. We write $\alpha\le\beta$ to mean $\alpha_i\le\beta_i$ for all i. We write e_k for the sequence whose k-th element is 1 and all other ones 0. We usually omit trailing zeros. For sequences α, β, and $\delta\ge 0$, let $\gamma(L,\beta,\delta)=\dim|L|-HL+|\beta|-\delta$.

The relative refined Severi degrees $N^{(S,L),\delta}(\alpha,\beta)(y)$ is defined in [BG, Def. 7.2]. Here $N^{(S,L),\delta}(\alpha,\beta)(1)$ is the relative Severi degree, i.e. the number of δ-nodal curves in $|L|$ not containing H, through $\gamma(L,\beta,\delta)$ general points, and with α_k given points of contact of order k with H, and β_k arbitrary points of contact of order k with H. By definition the relative refined Severi degrees contain the refined Severi degrees as special case: $N^{(S,L),\delta}(0,(LH))(y)=N^{(S,L),\delta}(y)$.

Theorem 8. ([BG, Thm. 7.5]) *Let L be a line bundle on S and let α, β be sequences with $I\alpha+I\beta=HL$, and let $\delta\ge0$ be an integer. If $\gamma(L,\beta,\delta)>0$, then*

$$\begin{aligned}N^{(S,L),\delta}(\alpha,\beta)(y)&=\sum_{k:\beta_k>0}[k]_y\cdot N^{(S,L),\delta}(\alpha+e_k,\beta-e_k)(y)\\&\quad+\sum_{\alpha',\beta',\delta'}\left(\prod_i[i]_y^{\beta_i'-\beta_i}\right)\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}N^{(S,L-H),\delta'}(\alpha',\beta')(y).\end{aligned}\tag{2.1}$$

Here the second sum runs through all α',β',δ' satisfying the condition

$$\begin{aligned}&\alpha'\le\alpha,\ \beta'\ge\beta,\ I\alpha'+I\beta'=H(L-H),\\&\delta'=\delta+g(L-H)-g(L)+|\beta'-\beta|-1=\delta-H(L-H)+|\beta'-\beta|.\end{aligned}\tag{2.2}$$

Initial conditions: *if $\gamma(L,\beta,\delta)=0$ we have $N^{(S,L),\delta}(\alpha,\beta)(y)=0$, except for $N^{(\mathbb{P}^2,H),0}((1),(0))(y)=1$, $N^{(\mathbb{P}(1,1,m),H),0}((1),(0))(y)=1$ and $N^{(\Sigma_m,kF),0}((k),(0))(y)=1$, for all $k\ge0$.*

2.3. **Long edge graphs.** We review long edge graphs from [BCK], [L], [LO], working in the context of refined invariants. They are very close related to Floor diagrams. We follow the presentation in [L], [LO]. The arguments used are similar to those of [L], [LO].

Definition 9. A *long edge graph* G is a graph (V,E) with a weight function $w:E\to\mathbb{Z}_{>0}$ satisfying the following.

(1) The vertex set is $V=\mathbb{Z}_{\ge0}$, the edge set E is finite.
(2) G can have multiple edges, but no loops.
(3) G has no short edges, i.e. no edges connecting i and $i+1$ of weight 1.

An edge connecting i and j with $i<j$ will be denoted $(i\to j)$ (note that there can be more than one such edge). The *length* of an edge $e=(i\to j)$ is $\ell(e):=j-i$.

Definition 10. Given a long edge graph $G=(V,E,w)$, the *refined multiplicity* of G is

$$M(G)(y):=\prod_{e\in E}([w(e)]_y)^2.$$

The *Severi multiplicity* $m(G)$ and the *Welschinger multiplicity* of G are

$$m(G) := M(G)(1) = \prod_{e\in E} w(e)^2, \qquad r(G) := M(G)(-1) = \begin{cases} 1 & \text{all } w(e) \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}$$

The *cogenus* of G is $\delta(G) := \sum_{e\in E}(\ell(e)w(e) - 1)$.

We denote $\operatorname{minv}(G)$ (resp. $\operatorname{maxv}(G)$) the smallest (resp. largest) vertex i of G adjacent to an edge. The *length* of G is $l(G) := \operatorname{maxv}(G) - \operatorname{minv}(G)$.

We denote $G_{(k)}$ the graph obtained by shifting all edges of G to the right by k.

Definition 11. Let G be a long edge graph. For any $j \in \mathbb{Z}_{\geq 0}$ let $\lambda_j(G) := \sum_e w(e)$, for e running through the edges $(i \to k)$ with $i < j \leq k$.

For $\beta = (\beta_0, \ldots, \beta_M)$ a sequence of nonnegative integers, G is called β-*allowable* if $\operatorname{maxv}(G) \leq M + 1$ and $\beta_{j-1} \geq \lambda_j(G)$ for all $j = 1, \ldots, M + 1$. G is called *strictly* β-*allowable* if it is β-allowable and furthermore all edges incident to 0 or $M + 1$ have weight 1. Also write $\overline{\lambda}_j(G) := \lambda_j(G) - \#\{\text{edges } (j - 1 \to j)\}$. G is called β-*semiallowable* if $\operatorname{maxv}(G) \leq M + 1$ and $\beta_{j-1} \geq \overline{\lambda}_j(G)$ for all j.

In this paper we will mostly consider the following sequences.

Notation 12. Let $c, d, m \in \mathbb{Z}_{\geq 0}$. We put $s(c, m, d) := (e_0, \ldots, e_d)$ with $e_i = c + mi$.

Definition 13. A long edge graph Γ is a *template* if for any vertex $1 \leq i \leq \ell(\Gamma) - 1$ there exists at least one edge $(j \to k)$ with $j < i < k$. A long edge graph G is called a *shifted template* if $G = \Gamma_{(k)}$ for some template $k \in \mathbb{Z}_{\geq 0}$.

Definition 14. Let G be β-allowable for $\beta = (\beta_0, \ldots, \beta_M)$. Define a new graph $\operatorname{ext}_\beta(G)$ by adding $\beta_{j-1} - \lambda_j(G)$ edges of weight 1 connecting $j - 1$ and j for all $j = 1, \ldots, M + 1$.

A β-*extended ordering* of G is a total ordering of the vertices and edges of $\operatorname{ext}_\beta(G)$, such that

(1) it extends the natural ordering of the vertices $0, 1, 2, \ldots$,
(2) if an edge e connects vertices i and j, then e is between i and j.

Two extended orderings o, o' of G are considered equivalent if there is an automorphism of the edges, permuting only edges connecting the same vertices and of the same weight which sends o to o'.

Definition 15. For a long edge graph let $P_\beta(G)$ be the number of β-extended orderings of G up to equivalence. Here $P_\beta(G)$ is defined to be 0, if G is not β-allowable. Furthermore let $P^s_\beta(G) := \begin{cases} P_\beta(G) & G \text{ strictly } \beta\text{-allowable,} \\ 0 & \text{otherwise.} \end{cases}$

Definition 16. Given $\beta \in \mathbb{Z}_{\geq 0}^{M+1}$, define

$$N^\delta_\beta(y) := \sum_G M(G) P^s_\beta(G), \quad n^\delta_\beta := \sum_G m(G) P^s_\beta(G), \quad W^\delta_\beta := \sum_G r(G) P^s_\beta(G),$$

where the summation is over all long edge graphs G of cogenus δ.

Notation 17. We denote $\Sigma_m := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m))$ the m-th rational ruled surface. Let F be the class of the fibre of the ruling and let E be the class of a section with $E^2 = -m$. We denote $H := E + mF$.

The connection to the refined Severi and tropical Welschinger numbers is given by

Theorem 18. *(1) For the refined Severi degrees of $\mathbb{P}^2$, $\mathbb{P}(1,1,m)$ and Σ_m we have $N^{d,\delta}(y) = N^\delta_{s(0,d,1)}(y)$, $N^{(\mathbb{P}(1,1,m),dH),\delta}(y) = N^\delta_{s(0,m,d)}(y)$, $N^{(\Sigma_m,cF+dH),\delta}(y) = N^\delta_{s(c,m,d)}(y)$.*

(2) For the Severi degrees we have $n^{d,\delta} = n^\delta_{s(0,d,1)}$, $n^{(\mathbb{P}(1,1,m),dH),\delta} = n^\delta_{s(0,m,d)}$, $n^{(\Sigma_m,cF+dH),\delta} = n^\delta_{s(c,m,d)}$.

(3) For the Welschinger numbers we have $W^{d,\delta} = W^\delta_{s(0,d,1)}$, $W^{(\mathbb{P}(1,1,m),dH),\delta} = W^\delta_{s(0,m,d)}$, $W^{(\Sigma_m,cF+dH),\delta} = W^\delta_{s(c,m,d)}$.

Proof. The proof is similar to that of [BCK, Thm. 2.7], we include it for completeness. It is enough to prove (1), because by Definition 16 and Definition 10 we have $n^\delta_\beta = N^\delta_\beta(1)$ and $W^\delta_\beta = N^\delta_\beta(-1)$, and we know $N^{(S,L),\delta}(1) = n^{(S,L),\delta}$, $N^{(S,L),\delta}(-1) = W^{(S,L),\delta}$ for any pair (S, L) of toric surface and toric line bundle. Furthermore it is enough to prove (1) in case $S = \Sigma_m$, because by Theorem 7 we have $N^{(\mathbb{P}(1,1,m),dH),\delta}(y) = N^{(\Sigma(1,1,m),dH),\delta}(y)$.

Let $\Delta = \Delta_{c,m,d}$ for $c, m, d \in \mathbb{Z}_{\geq 0}$. Let $\beta := s(c, m, d)$. We will show that N^δ_β is equal to the right hand side of Theorem 7, thus finishing the proof. First we show that there is a bijection between Δ-floor diagrams and strictly β-allowable long-edge graphs which respects the cogenus, by showing that both are in bijection to another set of graphs, which for the moment we will call β-graphs. A β-graph is defined precisely like a long edge graph, except that we also allow for short edges $(i \to i + 1)$ of weight 1, and we require $\beta_{j-1} = \lambda_j(G)$ for $j = 1, \dots, d + 1$, where as before $\lambda_j(G) = \sum_e w(e)$, with e running through the edges $(i \to k)$ with $i < j \leq k$. By definition it is clear that the map $G \mapsto \mathrm{ext}_\beta(G))$ defines a bijection from the strictly β-allowable long-edge graphs to the β-graphs, and the inverse is given by removing all short edges $(i \to i + 1)$ of weight 1 from a β-graph. We define the cogenus of a β-graph by $\delta(G) = \sum_e (l(e)w(e) - 1)$, with e running over all edges of G. It is obvious that $\delta(G) = \delta(\mathrm{ext}_\beta(G))$.

If $\mathcal{D}$ is a Δ-floor diagram, we first perform steps (1) and (2) in Definition 6. Then we identify all vertices we have created in step (1) to a vertex 0, and we identify all vertices we have created in step (2) to a vertex $d + 1$, in addition we add vertices $\mathbb{Z}_{\geq d+2}$ to the graph obtained this way. It is easy to see that in this way we get a β-graph $G(\mathcal{D})$. Clearly the map $\mathcal{D} \mapsto G(\mathcal{D})$ is injective, as all the steps are injective, and by definition is is also clear that it is surjective. If $\widetilde{\mathcal{D}}$ is a marking of $\mathcal{D}$, then we see that the total number of vertices of $\widetilde{\mathcal{D}}$ is equal to $d + \#E$ where E is the set of edges of $G(\mathcal{D})$. Defining $M(F) := \prod_e [w(e)]^2_y$ with e running through the edges of the β-graph F, Definitions 10 and 5 imply $\mathrm{mult}(\mathcal{D}) = M(G(\mathcal{D}))$ for a floor diagram $\mathcal{D}$ and $M(G) = M(\mathrm{ext}_\beta(G))$ for a long edge graph G. From the definitions we also see that

$$\delta(G(\mathcal{D})) = \sum_{e\in E} w(e)l(e) - \#E = \sum_{i=1}^{d} \lambda_i(G(\mathcal{D})) - \#E = \#(\Delta\cap\mathbb{Z}^2) - d - \#E - 1 = \delta(\widetilde{\mathcal{D}}).$$

Note that the markings of the Δ-floor diagram $\mathcal{D}$ are in bijection with the number of diagrams obtained by putting one vertex on every edge of $G(\mathcal{D})$ and ordering all the vertices of the new diagram, preserving the order of the vertices of $G(\mathcal{D})$, and such that the vertex introduced on an edge $(i \to j)$ lies between i and j. But this number clearly is the same as the number of linear orders on the union of the vertices and edges of $G(\mathcal{D})$, again preserving the order of the vertices and and such that the edge $(i \to j)$ lies between i and j. By definition this is just the number of β-extended orderings of the long edge graph corresponding to $\mathcal{D}$.

□

Remark 19. More generally the methods of [BG] will show (using also the notations from [LO]) the following refined version of [LO, Thm. 2.12] (see [BG, Rem. 5.8]).

(1) For any $\delta \geq 0$, any h-transversal lattice polygon the refined Severi degree is

$$N^{\Delta,\delta}(y) = \sum_{(\mathbf{l},\mathbf{r})} N^{\delta-\delta(\mathbf{l},\mathbf{r})}_{\beta(d^t,\mathbf{r}-\mathbf{l})}(y).$$

Here the summation is over all reorderings $\mathbf{l}$ and $\mathbf{r}$ of the multisets of left and right directions of Δ, satisfying $\delta(\mathbf{l},\mathbf{r}) \leq \delta$, $\beta(d^t,\mathbf{r}-\mathbf{l}) \in \mathbb{Z}^{M+1}_{\geq 0}$.

(2) With the same index of summation we have

$$n^{\Delta,\delta} = \sum_{(\mathbf{l},\mathbf{r})} n^{\delta-\delta(\mathbf{l},\mathbf{r})}_{\beta(d^t,\mathbf{r}-\mathbf{l})}, \quad W^{\Delta,\delta} = \sum_{(\mathbf{l},\mathbf{r})} W^{\delta-\delta(\mathbf{l},\mathbf{r})}_{\beta(d^t,\mathbf{r}-\mathbf{l})},$$

Following [L],[LO], we consider logarithmic versions of $P_\beta(G)$ and $P^s_\beta(G)$,

Definition 20. A *partition* of a long edge graph $G = (V, E, w)$ is a tuple $(G_1, \dots, G_n)$ of nonempty long edge graphs such that the disjoint union of the (weighted) edge sets of $G_1, \dots, G_n$ is the (weighted) edge set of G.

For any long edge graph define

$$\Phi_\beta(G) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{G_1,\dots,G_n} \prod_{j=1}^{n} P_\beta(G_j),$$

$$\Phi^s_\beta(G) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{G_1,\dots,G_n} \prod_{j=1}^{n} P^s_\beta(G_j),$$

where both summations are over the partitions of G.

Let

$$\mathcal{N}(\beta, y, t) := 1 + \sum_{\delta>0} N^\delta_\beta(y)t^\delta, \quad \mathcal{Q}(\beta, y, t) := \log(\mathcal{N}(\beta, y, t)) = \sum_{\delta>0} Q^\delta_\beta(y)t^\delta.$$

Then the same arguments as in [LO] show that

$$Q^{\delta}_{\beta}(y)=\sum_{G} M(G)\Phi^{s}_{\beta}(G), \tag{2.3}$$

where the summation is again over all long-edge graphs of cogenus δ.

Definition 21. Let G be a long edge graph. Let $\epsilon_0(G):=1$, if all edges adjacent to $\mathrm{minv}(G)$ have weight 1, and $\epsilon_0(G):=0$ otherwise. Similarly let $\epsilon_1(G):=1$, if all edges adjacent to $\mathrm{maxv}(G)$ have weight 1, and $\epsilon_1(G):=0$ otherwise.

By [L, Lem. 2.15] we have $\Phi^s_\beta(G)=0$, if G is not a shifted template. On the other hand [L, Cor. 3.5] says that for a template Γ we have

$$\Phi^s_\beta(\Gamma_{(k)})=\begin{cases}\Phi_\beta(\Gamma_{(k)}) & 1-\epsilon_0(\Gamma)\le k\le M+\epsilon_1(\Gamma)-\ell(\Gamma)\\ 0 & \text{otherwise.}\end{cases}$$

Together with (2.3), this gives the following refined version of [LO, Cor. 3.6].

Corollary 22. *Let $\beta=(\beta_0,\dots,\beta_M)\in\mathbb{Z}^{M+1}_{\ge 0}$. Then*

$$Q^{\delta}_{\beta}(y)=\sum_{\Gamma} M(\Gamma)\sum_{k=1-\epsilon_0(\Gamma)}^{M-\ell(\Gamma)+\epsilon_1(\Gamma)}\Phi_\beta(\Gamma_{(k)}),$$

where the first sum runs over all templates Γ of cogenus δ.

Theorem 23. [LO, Thm. 3.8] *Let G be a long edge graph. There exists a linear multivariate function $\Phi(G,\beta)$ in β, such that for any β such that G is β-semiallowable, we have $\Phi_\beta(G)=\Phi(G,\beta)$. Furthermore writing $\beta=(\beta_0,\dots,\beta_M)\in\mathbb{Z}^{M+1}_{\ge 0}$, the linear function $\Phi(G,\beta)$ is a linear combination of the β_i with $\mathrm{minv}(G)\le i\le\mathrm{maxv}(G)$.*

3. Multiplicativity theorems

In this section we will show that the generating functions for the refined Severi degrees on weighted projective spaces and rational ruled surfaces are multiplicative.

Theorem 24. *(1) Let $c\ge\delta$ and $d\ge\delta$, then $Q^{(\Sigma_m,cF+dH),\delta}(y)$ is a $\mathbb{Q}[y^{\pm1}]$-linear combination of 1, c, d, cd, m, md, md^2.*

(2) In particular if $c\ge\delta$, $d\ge\delta$, then $Q^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y)$ is a $\mathbb{Q}[y^{\pm1}]$-linear combination of 1, $c+d$, cd.

(3) Fix $m\ge 0$, $c\ge 0$. If $d\ge\delta$ then $Q^{(\Sigma_m,dH+cF),\delta}(y)$ is a polynomial of degree 2 in d.

(4) Fix $m\ge 0$. If $d\ge\delta$, then $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ is a polynomial of degree 2 in d. In particular for $d\ge\delta$, $Q^{d,\delta}(y)$ is a polynomial of degree 2 in d.

(5) If $d,m\ge\delta$, then $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ is a $\mathbb{Q}[y^{\pm1}]$-linear combination of 1, m, d, dm, d^2m.

Proof. (1) By Corollary 22 and Theorem 18, we have

$$Q^{(\Sigma_m,cF+dH),\delta}(y)=Q^{\delta}_{s(c,m,d)}(y)=\sum_{\Gamma}M(\Gamma)\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)}\Phi_{s(c,m,d)}(\Gamma_{(k)}),\tag{3.1}$$

with Γ running through all templates of cogenus δ.

Let Γ now be a template of cogenus δ, and let k be an integer in $[1-\epsilon_0(\Gamma),d-\ell(\Gamma)+\epsilon_1(\Gamma)]$. Then by definition we get $\Phi_{s(c,m,d)}(\Gamma_{(k)})=\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$. On the other hand by [LO, Lem. 4.2] we have $\overline{\lambda}_i(\Gamma)\le\delta$ for all i. By our assumption we have $c\ge\delta\ge\overline{\lambda}_i(\Gamma)$, thus Γ is $s(c+km,m,\ell(\Gamma)-1)$-semiallowable. Therefore $\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$ is a linear function in the $c+lm$, $k\le l\le k+\ell(\Gamma)-1$, thus it is linear function in c and km of the form $\alpha+\beta(c+km)+\gamma m$, with $\alpha,\beta,\gamma\in\mathbb{Q}$.

Let $M_1:=d-\ell(\Gamma)+\epsilon_1(\Gamma)+\epsilon_0(\Gamma)$, $M_2:=d-\ell(\Gamma)+\epsilon_1(\Gamma)-\epsilon_0(\Gamma)+1$. It is easy to see (and was already used in [L]) that for a template Γ of cogenus δ we have $\ell(\Gamma)-\epsilon_1(\Gamma)\le\delta$, so, by our assumption $d\ge\delta$, we have $M_1\ge 0$. Recall that for integers $b\ge a-1$ we have the trivial identity

$$\sum_{k=a}^{b}k=\frac{(a+b)(b-a+1)}{2}.$$

Thus we get

$$\begin{aligned}\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)}\Phi_{s(c,m,d)}(\Gamma_{(k)})&=\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)}\big(\alpha+\beta(c+km)+\gamma m\big)\\&=M_1(\alpha+\beta c+\gamma m)+\frac{M_1M_2}{2}\beta m,\end{aligned}$$

which is a $\mathbb{Q}$-linear combination of $1,c,d,cd,m,md,md^2$. Thus the claim follows by (3.1).

(2) By (1) $Q^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y)$ is a linear combination of 1, c, d, cd. It is clearly symmetric under exchange of c and d, and thus a linear combination of 1, $c+d$, cd.

(3) By Corollary 22 and Theorem 18,

$$Q^{(\Sigma_m,cF+dH),\delta}(y)=Q^{\delta}_{s(c,m,d)}(y)=\sum_{\Gamma}M(\Gamma)\sum_{k=1-\epsilon_o(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)}\Phi_{s(c,m,d)}(\Gamma_{(k)}),\tag{3.2}$$

with Γ running through all templates of cogenus δ.

Let Γ be a template of cogenus δ, and let k be an integer in $[1-\epsilon_0(\Gamma),d-\ell(\Gamma)+\epsilon_1(\Gamma)]$. Then by definition we get $\Phi_{s(c,m,d)}(\Gamma_{(k)})=\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$. For a rational number a we denote by $\lceil a\rceil$ the smallest integer bigger or equal to a. We put

$$k_{min}:=\max\left(1,\max\left(\left\lceil\frac{\overline{\lambda}_i(\Gamma)}{m}\right\rceil-i+1\,\middle|\,i=1,\dots,\ell(\Gamma)\right)\right).$$

For $k\ge k_{min}$ we have that $(k+i-1)m+c\ge\overline{\lambda}_i(\Gamma)$ for all i, thus Γ is $s(c+km,m,\ell(\Gamma)-1)$-semiallowable. Thus for $k\ge k_{min}$, we have that $\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$ is a linear function in the lm, $k\le l\le k+\ell(\Gamma)-1$, thus it is a linear function $\alpha+\beta km+\gamma m$, with $\alpha,\beta,\gamma\in\mathbb{Q}$.

By [LO, Lem. 4.2], we have $\overline{\lambda}_i(\Gamma) \le \delta - \ell(\Gamma) + i + \epsilon_1(\Gamma)$. As $\overline{\lambda}_i(\Gamma) \ge 0$, this implies

$$\left\lceil \frac{\overline{\lambda}_i(\Gamma)}{m} \right\rceil - i + 1 \le \delta + \epsilon_1(\Gamma) - \ell(\Gamma) + 1$$

for all i. By the inequality $\ell(\Gamma) - \epsilon_1(\Gamma) \le \delta$, already used in part (1), this implies $k_{min} \le \delta + \epsilon_1(\Gamma) - \ell(\Gamma) + 1$. By our assumption $d \ge \delta$, we have $d - \ell(\Gamma) + \epsilon_1(\Gamma) - k_{min} + 1 \ge 0$. Therefore the same argument as in (1) shows that the sum

$$\sigma(\Gamma, k_{min}) := \sum_{k=k_{min}}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)})$$

is a $\mathbb{Q}$-linear combination of 1, d, m, md, md^2. If we fix m, it is a linear combination of 1, d, d^2. But

$$\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c+km,m,l(\Gamma)-1)}(\Gamma) = \sigma(\Gamma, k_{min}) + \sum_{k=1-\epsilon_0(\Gamma)}^{k_{min}-1} \Phi_{s(c+km,m,l(\Gamma)-1)}(\Gamma).$$

The second sum is for fixed m just a finite number, thus the claim follows.

(4) As $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{(\Sigma_m,dH),\delta}(y)$, (4) is a special case of (3).

(5) By Corollary 22 and Theorem 18,

$$Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y) = \sum_{\Gamma} M(\Gamma) \sum_{k=1}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(0,m,d)}(\Gamma_{(k)}), \tag{3.3}$$

with Γ running through all templates of cogenus δ. According to Corollary 22, the inner sum starts at $k = 1-\epsilon_0(\Gamma)$. But Γ is a template and therefore not $(0, m, d)$-semiallowable. Thus (in case $\epsilon_0(\Gamma) = 1$), the contribution for $k = 0$ vanishes.

We have $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y)$, which is computed by the case $c = 0$ of (3.3). If $m \ge \delta$, then $k_{min} = 1$ for all templates Γ of cogenus δ, thus

$$Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y) = \sum_{\Gamma} M(\Gamma)\sigma(\Gamma, 1),$$

with Γ again running through the templates of cogenus δ. By (3) this is a $Q[y^{\pm 1}]$-linear combination of 1, d, m, md, md^2. $\square$

4. Relation to the conjectural generating functions of the refined invariants

In [GS] refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ of pairs (S, L) of a smooth projective surface and a line bundle on S were introduced. These are symmetric Laurent polynomials in a variable y, whose coefficients can be expressed universally (independent of S and L) as polynomials in the four intersection numbers L^2, LK_S K_S^2 and $c_2(S)$ on the surface. For toric surfaces S and sufficiently ample line bundles L the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ and refined Severi degrees $N^{(S,L),\delta}(y)$ are conjectured to agree ([GS, Conj. 80]).

Conjecture 25. *Let (S, L) be a pair of a smooth toric surface and a line bundle on L.*

(1) If L is δ-very ample on S, then $\widetilde{N}^{(S,L),\delta}(y) = N^{(S,L),\delta}(y)$.
(2) $\widetilde{N}^{d,\delta}(y) = N^{d,\delta}(y)$ *for* $\delta \le 2d-2$.
(3) $\widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1,dH+cF),\delta}(y) = N^{(\mathbb{P}^1\times\mathbb{P}^1,dH+cF),\delta}(y)$ *for* $\delta \le \min(2d,2c)$.
(4) $\widetilde{N}^{(\Sigma_m,dH+cF),\delta}(y) = N^{(\Sigma_m,dH+cF),\delta}(y)$ *for* $\delta \le \min(2d,c)$.

Remark 26. Note that by definition, if Conjecture 25 is true, then $N_\delta(d;y) = \widetilde{N}^{d,\delta}(y)$ for all d, δ, and $N_\delta(\Sigma_m, dH+cF)(y) = \widetilde{N}^{(\Sigma_m,dH+cF),\delta}(y)$ for all m,d,c,δ. This is because both sides are polynomials in d (respectively d,c) with coefficients in $\mathbb{Q}[y]$, which coincide for all sufficiently large d (respectively for all sufficiently large d,c).

In [GS, Conj. 67] also a generating function for the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ is conjectured (and thus by Remark 26 for the $N_\delta(d;y)$ and the $N_\delta((\Sigma_m, cF+dH);y))$ [GS, Conj.67]. We list a number of equivalent formulations.

Notation 27. We start by introducing some notations about quasimodular forms and theta functions, and reviewing some standard facts, which we will use throughout the paper. Modular forms depend on a variable τ in the complex upper half plane, and have a Fourier development in terms of $q := e^{2\pi i\tau}$. We will write them as functions $f(q)$, because we are only interested in the coefficients of their Fourier development. Similarly theta functions will be written as functions $g(y,q)$, for $y = e^{2\pi i z}$, with $z \in \mathbb{C}$ and $q = e^{2\pi i\tau}$. The Eisenstein series

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n>0}\sum_{d|n} d^{2k-1}q^k$$

are for $2k \ge 4$ modular forms of weight $2k$ on $SL_2(\mathbb{Z})$, whereas $G_2(q)$ is only a quasimodular form of weight 2 on $SL_2(\mathbb{Z})$. The Dirichlet η-function and the discriminant $\Delta(q)$ are

$$\eta(q) := q^{1/24}\prod_{n>0}(1-q^n), \qquad \Delta(q) = \eta(q)^{24} = q\prod_{n>0}(1-q^n)^{24}.$$

The discriminant is a cusp form of weight 12 on $SL_2(\mathbb{Z})$. The operator $D := q\frac{\partial}{\partial q}$ sends (quasi)modular forms of weight $2k$ to quasimodular forms of weight $2k+2$. We denote two of the standard theta functions by

$$\theta(y) = \theta(y,q) := \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} y^{n+\frac{1}{2}} = q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})\prod_{n>0}(1-q^n)(1-q^n y)(1-q^n/y),$$
$$\theta_2(y,q) := \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} y^n,$$

and the theta zero value $\theta_2(q^2) := \theta_2(0,q^2) = \sum_{n\in\mathbb{Z}}(-1)^m q^{n^2} = \frac{\eta(q)^2}{\eta(q^2)}$. In addition to $D := q\frac{\partial}{\partial q}$ we also consider $' = y\frac{\partial}{\partial y}$. Let

$$\widetilde{\Delta}(y,q) := \frac{\eta(q)^{18}\theta(y)^2}{y-2+y^{-1}} = q\prod_{n=1}^{\infty}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2,$$

$$\widetilde{DG}_2(y,q) := \sum_{m=1}^\infty \sum_{d|m} \frac{m}{d}[d]_y^2 q^m, \quad D\widetilde{DG}_2(y,q) := \sum_{m=1}^\infty \sum_{d|m} \frac{m^2}{d}[d]_y^2 q^m.$$

Conjecture 28. *There exist universal power series $B_1(y,q)$, $B_2(y,q)$ in $\mathbb{Q}[y,y^{-1}][\![q]\!]$, such that for all pairs (S,L) of a smooth projective surface and a line bundle on L, we have*

$$\sum_{\delta\ge 0} \widetilde{N}^{(S,L),\delta}(y)(\widetilde{DG}_2)^\delta = \frac{(\widetilde{DG}_2/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q)/q^2)^{\chi(\mathcal{O}_S)/2}} \tag{4.1}$$

We give two equivalent reformulations. $\widetilde{DG}_2$ as a power series in q starts with q, let

$$g(t) := g(y,t) = t + ((-y^2-4y-1)/y)t^2 + ((y^4+14y^3+30y^2+14y+1)/y^2)t^3 + O(t^4)$$

be its compositional inverse. Write $g'(t) := \frac{\partial g}{\partial t}$.

Remark 29. Let $R \in \mathbb{Q}[y^{\pm 1}][[q]]$ be a formal power series. For polynomials $M^{(S,L),\delta}(y) \in \mathbb{Q}[y^{\pm 1}]$ the following three formulas are equivalent:

(1) $\displaystyle\sum_{\delta\ge 0} M^{(S,L),\delta}(y)(\widetilde{DG}_2)^\delta = \frac{(\widetilde{DG}_2/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q)/q^2)^{\chi(\mathcal{O}_S)/2}} R(y,q)$

(2) $\displaystyle\sum_{\delta\ge 0} M^{(S,L),\delta}(y)t^\delta = \frac{(t/g(t))^{\chi(L)} B_1(y,g(t))^{K_S^2}}{B_2(y,g(t))^{-LK_S}} \left(\frac{g(t)g'(t)}{\widetilde{\Delta}(y,g)}\right)^{\chi(\mathcal{O}_S)/2} R(y,g(t)),$

(3) For all $\delta \ge 0$

$$M^{(S,L),\delta}(y) = \operatorname*{Coeff}_{q^{(L^2-LK_S)/2}}\left[\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta} \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}} R(y,q)\right]$$

Proof. (2) is equivalent to (1) by noting that $D\widetilde{DG}_2(y,g(t)) = \frac{g(t)}{g'(t)}\frac{\partial \widetilde{DG}_2(y,g(t))}{\partial t} = \frac{g(t)}{g'(t)}$.

Let A be a commutative ring, and let $f \in A[[q]]$, $g \in q + qA[[q]]$. Then we get by the residue formula that

$$f(q) = \sum_{l=0}^\infty g(q)^l \operatorname*{Coeff}_{q^0}\left[\frac{f(q)Dg(q)}{g(q)^{l+1}}\right].$$

Applying this with $g(q) = \widetilde{DG}_2$ shows that (1) is equivalent to (3). □

Part (2) of Remark 29 shows in particular that according to Conjecture 28 the $N^{(S,L),\delta}(y)$ have a generating function of the form (1.1).

Remark 30. We will in the future use the formula (3) of Remark 29. Note that this also has the following interpretation. Write

$$A^{(S,L)}(y,q) := \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}}.$$

Then the refined count of curves in $|L|$ with only nodes as singularities satisfying k general point conditions is $\operatorname{Coeff}_{q^{(L(L-K_S)/2}}[\widetilde{DG}_2(y,q)^k A^{(S,L)}(y,q)]$. Thus it seems natural to expect the following general principle: To each condition c that we can impose at points of S to curves C in $|L|$ (e.g. C passing through a point with given multiplicity), or just to points in S, (e.g. S having a singular point) there corresponds a power series $L_c \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that, for L sufficiently ample, the refined count of curves in $|L|$ on S satisfying conditions $c_1, \ldots, c_n$ is $\mathrm{Coeff}_{q^{(L(L-K_S)/2}}[A^{(S,L)}(y,q)\prod_{i=1}^n L_{c_i}]$. According to this principle the power series corresponding to passing through a point of S would be $\widetilde{DG}_2$. In the second half of this paper we will give a number of instances of this principle.

By Remark 26 for $\mathbb{P}^2$ and rational ruled surfaces the conjecture says in particular

$$N_\delta(d;y) = \mathop{\mathrm{Coeff}}_{q^{(d^2+3d)/2}}\left[\widetilde{DG}_2(y,q)^{d(d+3)/2-\delta}\frac{B_1(y,q)^9}{B_2(y,q)^{3d}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\right] \tag{4.2}$$

$$N_\delta((\Sigma_m, cF+dH);y) = \tag{4.3}$$

$$\mathop{\mathrm{Coeff}}_{q^{(d+1)(c+1+md/2)-1}}\left[\frac{\widetilde{DG}_2(y,q)^{(d+1)(c+1+md/2)-1-\delta}B_1(y,q)^8}{B_2(y,q)^{2c+(m+2)d}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\right]$$

With $B_1(y,q)$, $B_2(y,g)$ given below modulo q^{18} we have the following corollary.

Corollary 31. *(1) The formula (4.2) and Conjecture 25(2) are true for $\delta \le 17$.*
(2) In case $m=0$ the formula (4.3) and Conjecture 25(2) is true for $\delta \le 12$.
(3) The formula (4.3) and Conjecture 25(3) are true for all m and $\delta \le 8$.

Proof. (1). Using the Caporaso-Harris recursion, we computed the $N^{d,\delta}(y)$ for $d \le 19$, $\delta \le 19$. This also computes the $Q^{d,\delta}$ for $d \le 19$, $\delta \le 19$. Part (4) of Theorem 24 gives $Q^{d,\delta} = Q_\delta(d)$ for $d \ge \delta$. As $Q_\delta(d;y)$ is a polynomial of degree 2 in d, the computation above determines $Q_\delta(d;y)$ and thus the $N_\delta(y;d)$ for $\delta \le 17$, giving the claim.

(2) and (3). Using again the Caporaso-Harris recursion we computed the $N^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ for $c,d \le 13, \delta \le 13$. Again this gives the $Q^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}$ for $c,d \le 13$, $\delta \le 13$. By part (2) of Theorem 24 we have that $Q^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta} = Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ for $c,d \ge \delta$. As $Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ is a polynomial of bidegree $(1,1)$ in c,d, the computation above determines $Q_\delta(((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ and thus the $N_\delta((\Sigma_0, cF+dH);y)$ for $\delta \le 12$. As $Q_\delta((\Sigma_m, cF+dH);y)$ is a linear combination of 1, c, cd, m, md, md^2, in order to prove (2) we only need to determine the coefficients of m, md, md^2. For this we can restrict to the case $m=1$, We computed $N^{(\Sigma_1,cF+dH),\delta}(y)$ for $c, \le 9$, $d \le 10$. This determines the coefficients of m, md, md^2 of $Q_\delta((\Sigma_m, cF+dH);y)$ for $\delta \le 8$, giving the claim. $\square$

We list the leading terms of $B_1(y,q)$ and $B_2(y,q)$, with omitted terms determined by symmetry.

$$B_1(y,q) = 1 - q - (y+3+1/y)q^2 + (y^2+10y+17+\ldots)q^3 - (18y^2+87y+135+\ldots)q^4$$
$$+ (12y^3+210y^2+728y+1061+\ldots)q^5 - (2y^4+259y^3+2102y^2+5952y+8236+\ldots)q^6$$
$$+ (162y^4+3606y^3+19668y^2+48317y+64253+\ldots)q^7 - (47y^5+3789y^4+41999y^3+177800y^2$$
$$+ 392361y+505678+\ldots)q^8 + (5y^6+2416y^5+60202y^4+445989y^3+1576410y^2+3197831y$$

$$
\begin{aligned}
&+ 4018919 + \dots)q^9 - (896y^6 + 58504y^5 + 793194y^4 + 4483755y^3 + 13818256y^2 + 26192369y\\
&+ 32243357 + \dots)q^{10} + (176y^7 + 38236y^6 + 1017512y^5 + 9382867y^4 + 43520558y^3 + 120325637y^2\\
&+ 215688799y + 260959201 + \dots)q^{11} - (14y^8 + 16393y^7 + 944954y^6 + 14738959y^5 + 103623419y^4\\
&+ 412518547y^3 + 1043940859y^2 + 1785764779y + 2129062780 + \dots)q^{12} + (4384y^8 + 631224y^7\\
&+ 17534642y^6 + 190488676y^5 + 1092093647y^4 + 3845977628y^3 + 9041155627y^2 + 14862430058y\\
&+ 17497499443 + \dots)q^{13} - (658y^9 + 298228y^8 + 15816382y^7 - 273455570y^6 + 2279829046y^5\\
&+ 11131917064y^4 + 35435770399y^3 + 78257451025y^2 + 124310761787y + 144758147754 + \dots)q^{14}\\
&+ (42y^{10} + 96604y^9 + 10758628y^8 + 308060184y^7 + 3800583626y^6 + 25834889754y^5\\
&+ 110712006552y^4 + 323710356925y^3 + 677516096371y^2 + 1044598390812y + 1204824660925 + \dots)q^{15}\\
&- (20284y^{10} + 5452043y^9 + 272316274y^8 + 5094738491y^7 + 48707795806y^6 + 281165238614y^5\\
&+ 1080786159810y^4 + 2938608835049y^3 + 5869829083826y^2 + 8816117002571y + 10082791437552 + \dots)q^{16}\\
&+ (2472y^{11} + 2015609y^{10} + 188032406y^9 + 5506997958y^8 + 75206548205y^7 + 588088410636y^6\\
&+ 2967196356618y^5 + 10400483736235y^4 + 26552849592007y^3 + 50907878544033y^2 + 74707191955540y\\
&+ 84801344804750 + \dots)q^{17} + O(q^{18}),
\end{aligned}
$$

$$
\begin{aligned}
&B_2(y,q) = \frac{1}{(1-yq)(1-q/y)}\big(1 + 3q - (3y + 1 + 3/y)q^2 + (y^2 + 8y + 18 + \dots)q^3\\
&- (13y^2 + 53y + 76 + \dots)q^4 + (7y^3 + 100y^2 + 316y + 455 + \dots)q^5 - (y^4 + 112y^3 + 779y^2\\
&+ 2076y + 2819 + \dots)q^6 + (67y^4 + 1243y^3 + 6129y^2 + 14386y + 18870 + \dots)q^7 - (19y^5\\
&+ 1281y^4 + 12417y^3 + 48879y^2 + 104034y + 132579 + \dots)q^8 + (2y^6 + 822y^5 + 17542y^4\\
&+ 117829y^3 + 393703y^2 + 775411y + 965540 + \dots)q^9 - (310y^6 + 17206y^5 + 207074y^4\\
&+ 1085712y^3 + 3197506y^2 + 5913778y + 7223539 + \dots)q^{10} + (62y^7 + 11505y^6 + 267658y^5\\
&+ 2249872y^4 + 9825927y^3 + 26163595y^2 + 45935572y + 55208836 + \dots)q^{11} - (5y^8 + 5076y^7\\
&+ 253785y^6 + 3555348y^5 + 23210920y^4 + 87929247y^3 + 215557414y^3 + 362229349y\\
&+ 429395117 + \dots)q^{12} + (1397y^8 + 174456y^7 + 4304488y^6 + 42877083y^5 + 231296838y^4\\
&+ 781220881y^3 + 1787129788y^2 + 2892830316y + 3388742192 + \dots)q^{13} - (215y^9 + 85117y^8\\
&+ 3983060y^7 + 62465678y^6 + 484877903y^5 + 2249516882y^4 + 6909207376y^3 + 14901830113y^2\\
&+ 23353834274y + 27076007072 + \dots)q^{14} + (14y^{10} + 28472y^9 + 2793096y^8 + 71942817y^7\\
&+ 818536892y^6 + 5240193024y^5 + 21495922606y^4 + 60931593665y^3 + 124910088474y^2\\
&+ 190304808803y + 218642432495 + \dots)q^{15} - (6158y^{10} + 1462435y^9 + 65354234y^8\\
&+ 1118442331y^7 + 9987960061y^6 + 54777796045y^5 + 202738958803y^4 + 536439701989y^3\\
&+ 1052049129591y^2 + 1563445962327y + 1781883877192 + \dots)q^{16} + (770y^{11} + 558612y^{10}\\
&+ 46524657y^9 + 1238412474y^8 + 15681201140y^7 + 115681622517y^6 + 558367283967y^5\\
&+ 1893273288345y^4 + 4718572145488y^3 + 8899835406922y^2 + 12937087920811y\\
&+ 14639451592197 + \dots)q^{17} + O(q^{18})\big).
\end{aligned}
$$

As noted above, the refined Severi degrees $N^{(S,L),\delta}(y)$ specialize at $y=1$ to the tropical Welschinger numbers $W^{(S,L),\delta}$. We specialize the above conjectures of [GS] to the tropical Welschinger numbers. As the Caporaso-Harris recursion for the tropical Welschinger numbers is computationally much more efficient than that for the refined Severi degrees, the conjectures for the tropical Welschinger numbers can be proven for much higher δ.

Let $\eta(q) := q^{1/24}\prod_{n>0}(1-q^n)$ the Dirichlet eta function, $G_2(q) := -\frac{1}{24}+\sum_{n>0}\sum_{d|n} dq^n$ be the Eisenstein series, and write

$$\overline{G}_2(q) := \widetilde{DG}_2(-1,q) = G_2(q) - G_2(q^2) = \sum_{n>0}\left(\sum_{d|n,\ d \text{ odd}} \frac{n}{d}\right) q^n.$$

We note that $\widetilde{DG_2}(-1,q) = \overline{G}_2(q)$, and $\widetilde{\Delta}(-1,q) = \eta(q)^{16}\eta(q^2)^4$. We write $\overline{B}_1(q) := B_1(-1,q)$, $\overline{B}_2(q) := B_2(-1,q)$. Conjecture 25 specializes to the following (see also [GS]).

Conjecture 32.

(4.4)

$$W_\delta(d) = \underset{q^{(d^3+3d)/2}}{\operatorname{Coeff}}\left[\overline{G}_2(q)^{d(d+3)/2-\delta}\frac{\overline{B}_1(q)^9(D\overline{G}_2(q))^{1/2}}{\overline{B}_2(q)^{3d}\eta(q)^8\eta(q^2)^2}\right],$$

(4.5)

$$W_\delta((\Sigma_m, cF+dH)) = \underset{q^{((d+1)(c+1+md/2)-1}}{\operatorname{Coeff}}\left[\frac{\overline{G}_2(q)^{(d+1)(c+1+md/2)-1-\delta}\overline{B}_1(q)^8(D\overline{G}_2(q))^{1/2}}{\overline{B}_2(q)^{2c+(m+2)d}\eta(q)^8\eta(q^2)^2}\right]$$

With $\overline{B}_1(q)$, $\overline{B}_2(q)$ given below modulo q^{31} we have the following corollary.

Corollary 33. *(1) The formula (4.4) is true for $\delta \le 30$. Furthermore for $\delta \le 30$ and $d \ge \delta/3+1$ we have $W^{d,\delta} = W_\delta(d)$.*

(2) On $\mathbb{P}^1\times\mathbb{P}^1$ the formula (4.5) is true for $\delta \le 20$. Furthermore for $\delta \le 20$ and $\delta \le \min(20, 3c, 3d)$, we have $W^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta} = W_\delta(\mathbb{P}_1\times\mathbb{P}_1, cF+dH)$.

(3) For $m>0$, the formula (4.5) is true for $\delta \le 11$. Furthermore for $\delta \le \min(11, 3d, c)$ we have $W^{(\Sigma_m, cF+dH),\delta} = W_\delta(\Sigma_m, cF+dH)$.

Proof. (1) Using the Caporaso-Harris recursion, we computed to the $W^{d,\delta}$ for $d \le 32$, $\delta \le 33$. This also computes the $Q^{d,\delta}(-1)$ for $d \le 32$, $\delta \le 33$. The same argument as in the proof of Corollary 31 shows (1). Using again the Caporaso-Harris recursion we computed the $W^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}$ for $c, d \le 21, \delta \le 22$, and computed $W^{(\Sigma_1, cF+dH),\delta}(y)$ for $c, d, \delta \le 13$. The same argument as in the proof of Corollary 31 gives (2) and (3). $\square$

$$\overline{B}_1(q) = 1 - q - q^2 - q^3 + 3q^4 + q^5 - 22q^6 + 67q^7 - 42q^8 - 319q^9 + 1207q^{10} - 1409q^{11}$$
$$- 3916q^{12} + 20871q^{13} - 34984q^{14} - 37195q^{15} + 343984q^{16} - 760804q^{17} - 81881q^{18}$$

$$+ 5390386q^{19} - 15355174q^{20} + 8697631q^{21} + 79048885q^{22} - 293748773q^{23} + 329255395q^{24}$$
$$+ 1041894580q^{25} - 5367429980q^{26} + 8780479642q^{27} + 10991380947q^{28}$$
$$- 93690763368q^{29} + 203324385877q^{30} + O(q^{31}),$$
$$\overline{B}_2(q) = 1 + q + 2q^2 - q^3 + 4q^4 + 2q^5 - 11q^6 + 24q^7 + 4q^8 - 122q^9 + 313q^{10} - 162q^{11}$$
$$- 1314q^{12} + 4532q^{13} - 4746q^{14} - 13943q^{15} + 68000q^{16} - 105786q^{17} - 124968q^{18}$$
$$+ 1025182q^{19} - 2139668q^{20} - 443505q^{21} + 15157596q^{22} - 41007212q^{23} + 19514894q^{24}$$
$$+ 214218876q^{25} - 755331892q^{26} + 780656576q^{27} + 2776494907q^{28}$$
$$- 13420432234q^{29} + 20749875130q^{30} + O(q^{31}).$$

5. Correction term for singularities

In this section we want to extend the above results and conjectures to surfaces with singularities. This section is partially motivated by the paper [LO], where this question is studied for the non-refined invariants for toric surfaces with rational double points. We have conjectured above and given evidence that there exist generating functions for the refined node polynomials on smooth toric surfaces S, of the form $A_1^{L^2}A_2^{LK_S}A_3^{K_S^2}A_4^{\chi(\mathcal{O}_S)}$ for universal power series $A_i \in \mathbb{Q}[y^{\pm1}][[q]]$. It seems natural to conjecture that this extends to singular surfaces in the following form: for every analytic type of singularities c there is a universal power series $F_c(y,q)$ and the generating function for a singular surface S is $A_1^{L^2}A_2^{LK_S}A_3^{K_S^2}A_4^{\chi(\mathcal{O}_S)}\prod_c F_c^{n_c}$, where n_c is the number of singularities of S of type c. For the case of toric surfaces given by h-transversal lattice polygons with only rational double points this problem has been solved in [LO] for the (non-refined) Severi degrees.

We start out by formulating a conjecture for general singular toric surfaces, and then give more precise results for specific singularities. For rational double points we conjecture that somewhat surprisingly the power series $F_c(y,q)$ is independent of y. In particular this says that the correction factor for A_n-singularities, determined in [LO] for the Severi degrees, is the same for the Severi degrees and the tropical Welschinger invariants.

Now let S be a normal toric surfaces. We want to formulate a conjecture about the refined Severi degrees $N^{(S,L),\delta}(y)$. Note that the tropical curves counted in $N^{(S,L),\delta}(y)$ are not required to pass through any of the singular points of S. One can also reformulate the same conjecture in terms of the minimal resolution of S, i.e. a resolution $\pi : \widehat{S} \to S$, which contains no (-1) curves in the fibres of π.

Conjecture 34. *For every analytic type of singularities c there are formal power series $F_c \in \mathbb{Q}[y^{\pm1}][[q]]$, $\widehat{F}_c \in \mathbb{Q}[y^{\pm1}][[q]]$ such that the following hold. Let (S,L) be a pair of a projective toric surface and a toric line bundle on S. Let $\widehat{S}$ be a minimal toric resolution of S and denote by L also the pullback of L to $\widehat{S}$. Define $N^{(\widehat{S},L),\delta}(y) := N^{(S,L),\delta}(y)$. If L*

is δ-very ample on S, then

(5.1)
$$N^{(S,L),\delta}(y)=\underset{q^{L(L-K_S)/2}}{\mathrm{Coeff}}\left[\frac{\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta}B_1(y,q)^{K_S^2}}{B_2(y,q)^{-LK_S}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_c F_c(y,q)^{n_c}\right],$$

(5.2)
$$N^{(\widehat{S},L),\delta}(y)=\underset{q^{L(L-K_{\widehat{S}})/2}}{\mathrm{Coeff}}\left[\frac{\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta}B_1(y,q)^{K_{\widehat{S}}^2}}{B_2(y,q)^{-LK_{\widehat{S}}}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_c \widehat{F}_c(y,q)^{n_c}\right].$$

Here c runs through the analytic types of singularities of S, and n_c is the number of singularities of S of type c.

We can see that the two formulas formulas (5.1), (5.2) are equivalent. Note that $LK_S = LK_{\widehat{S}}$. On the other hand it is easy to see that $K_{\widehat{S}}^2 = K_S^2 - \sum_c n_c e_c$ where e_c is a rational number depending only on the singularity type c. Thus the two formulas are equivalent, via the identification

$$\widehat{F}_c(y,q) = F_c(y,q)B_1(y,q)^{e_c}.$$

It turns out that the power series $\widehat{F}_c(y,q)$ are usually simpler, so we will restrict our attention to them. Note that for a rational double point c we have $e_c = 0$ and thus $F_c = \widehat{F}_c$.

We give a slightly more precise version of the conjecture for a weighted projective space $\mathbb{P}(1,1,m)$ and its minimal resolution Σ_m, and prove some special cases of it. In this case the exceptional divisor is the section E with self intersection $-m$. The weighted projective space $\mathbb{P}(1,1,m)$ has one singularity of type $\frac{1}{m}(1,1)$, i.e. the cyclic quotient of $\mathbb{C}^2$ by the m-th roots of unity μ_m acting by $\epsilon(x,y) = (\epsilon x, \epsilon y)$. We write c_m for this singularity. It is elementary to see that

$$K_{\Sigma_m} = -2H + (m-2)F = -\frac{m+2}{m}H - \frac{m-2}{m}E, \quad K_{\mathbb{P}(1,1,m)} = -\frac{m+2}{m}H,$$
$$e_{c_m} = \frac{(m-2)^2}{m}, \quad K_{\Sigma_m}^2 = 8, \quad dHK_{\Sigma_m} = d(m+2), \quad \chi(\Sigma_m, dH) = (md+2)(d+1)/2.$$

Conjecture 35. *If $\delta \le 2d-1$, then*

(5.3)
$$N^{(\Sigma_m,dH),\delta}(y)=\underset{q^{\frac{m}{2}d^2+(\frac{m}{2}+1)d}}{\mathrm{Coeff}}\left[\frac{\widetilde{DG}_2(y,q)^{\frac{m}{2}d^2+(\frac{m}{2}+1)d-\delta}B_1(y,q)^8}{B_2(y,q)^{d(m+2)}}\Big(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\Big)^{1/2}\widehat{F}_{c_m}(y,q)\right].$$

Furthermore we have for $m \ge 2$

$$\begin{aligned}\widehat{F}_{c_m} &= 1 - mq + ((m-2)y + (m^2/2 + 3m/2 - 5) + (m-2)y^{-1})q^2\\ &\quad - ((m^2+5m-14)y + (m^3+9m^2+44m-132)/6 + (m^2+5m-14)y^{-1})q^3 + O(q^4),\end{aligned}$$

and

$$\widehat{F}_{c_2} = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = 1-2q+2q^4-2q^9+\dots,$$

$$\begin{aligned}\widehat{F}_{c_3} &= 1-3q+(y+4+y^{-1})q^2-(10y+18+10y^{-1})q^3+((6y^2+70y+115+70y^{-1}+6y^{-2})q^4\\ &\quad -((y^3+94y^2+473y+721y+473y^{-1}+94y^{-2}+y^{-3})q^5+O(q^6)\\ \widehat{F}_{c_4} &= 1-4q+(2y+9+2y^{-1})q^2-(22y+42+22y^{-1})q^3\\ &\quad +((14y^2+164y+273+164y^{-1}+14y^{-2})q^4+O(q^5).\end{aligned}$$

Proposition 36. *Let $\delta_2=8$, $\delta_3=5$, $\delta_4=4$, $\delta_m=3$ for $m\ge 5$. Then (5.3) is correct for $m\ge 2$ and $\delta\le \min(\delta_m,d)$.*

Proof. Using the Caporaso Harris recursion we computed $N^{(\Sigma_m,dH),\delta}$ for $2\le m\le 4$, $\delta\le\delta_m$ and $d\le d_m$ with $d_2=10$, $d_3=7$, $d_4=6$. We find that in this range (5.3) holds for $\delta\le\min(2d-1,\delta_m)$. By part (3) of Theorem 24 we have that $Q^{(\Sigma_m,dH,\delta)}$ is a polynomial of degree 2 in d for $d\ge\delta$. By the computation we know this polynomial in the following cases: $(m=2,\ \delta\le 8)$, $(m=3,\ \delta\le 5)$, $(m=4,\ \delta\le 4)$. This shows the result for $m=2,3,4$. Finally by part (5) of Theorem 24 we have that $Q^{(\Sigma_m,dH,\delta)}(y)$ is for $d,m\ge\delta$ a polynomial in d and m of degree 2 in d and 1 in m. By the above we know this polynomial as a polynomial in d for $\delta=0,1,2,3$ and $m=3,4$. This determines it and thus also $Q^{(\Sigma_m,dH,\delta)}(y)$ and therefore also $N^{(\Sigma_m,dH,\delta)}(y)$, for $\delta=0,1,2,3$ and $d,m\ge\delta$. The result follows. □

The non-refined Severi degrees for toric surfaces with only rational double points given by h transversal lattice polygons have been studied in [LO]. The only rational double points which can occur in this case are A_n singularities. For such surfaces they prove the analogue of Conjecture 34 for $y=1$ with precise bounds. Furthermore they show

$$F_{a_n}(1,q)=\frac{\eta(q)^{n+1}}{\eta(q^{n+1})}=\prod_{k>0}\frac{(1-q^k)^{n+1}}{1-q^{(n+1)k}},$$

where we denote $F_{a_n}(y,q)$ the power series $F_c(y,q)$ for c an A_n singularity. We conjecture that the same result holds also for the refined Severi degrees with the $F_{a_n}(y,q)$ independent of y.

Conjecture 37. *Let S be projective normal toric surface with only rational double points, more precisely with n_k singularities of type A_k for all k (with n_k only nonzero for finitely many k). If L is δ-very ample on S, then*

$$N^{(S,L),\delta}(y)=\underset{q^{L(L-K_S)/2}}{\mathrm{Coeff}}\left[\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta}\frac{B_1(y,q)^{K^2_{\widehat{S}}}}{B_2(y,q)^{-LK_{\widehat{S}}}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_k\left(\frac{\eta(q)^{k+1}}{\eta(q^{k+1})}\right)^{n_k}\cdot\right].$$

Remark 38. (1) $\mathbb{P}(1,1,2)$ has an A_1 singularity, and as we saw Σ_2 is a resolution of $\mathbb{P}(1,1,2)$. It is standard that $\theta_2(2\tau)=\frac{\eta(\tau)^2}{\eta(2\tau)}$. Thus for $\mathbb{P}(1,1,2)$ Conjecture 37 is a special case of Conjecture 35 and Proposition 36 gives evidence for it.

(2) We also used a version of the Caporaso Harris recursion for $\mathbb{P}(1,2,3)$. With the line bundle dH with d small for H the hyperplane bundle. $\mathbb{P}(1,2,3)$ has one A_1 and one A_2 singularity, also in this case Conjecture 37 is confirmed in the realm considered.

(3) Note that the conjecture that the $F_{a_n}(y,q)$ are independent of y says in particular that the correction factor for the A_n singularities is the same for Severi degrees and tropical Welschinger invariants.

We want to generalise this conjecture in another direction. Let S be a singular toric surface with singular points $p_1,\ldots,p_r$ and a minimal toric resolution $\widehat{S}$ with exceptional divisors $E_1,\ldots,E_r$. Let L be a toric line bundle on S. We have seen that $N^{(\widehat{S},L),\delta}(y)=N^{(S,L),\delta}(y)$ is a refined count of δ-nodal curves on S, which are not required to pass through the singular locus of S. In a similar way we can interpret $N^{(\widehat{S},L-k_1E_1-\ldots-k_rE_r),\delta}(y)$ as a refined count of curves in $|L|$ on S which pass through the singular points p_i with multiplicity $-k_iE_i^2$. This even makes sense if L is only a class of Weil divisors on S, the k_i are not necessarily integral but $L-k_1E_1-\ldots-k_rE_r$ is a Cartier divisor on $\widehat{S}$. In this case the curves we count on S are Weil divisors.

Here we will consider this question only in the case that S has only A_1 singularities. Denote $\eta(q)=q^{1/24}\prod_{n>0}(1-q^n)$ the Dirichlet eta function. Let $\theta_2(q):=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2/2}$ be one of the standard theta functions. Recall the Jacobi triple product formula

$$\eta(q^2)^3=q^{1/4}\sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)}.$$

We define functions $f_l(q)$, for $l\in\mathbb{Z}_{\geq 0}$ by

$$\begin{aligned} f_{2k}(q)&=\frac{(-1)^k}{(2k)!}\sum_{n\in\mathbb{Z}}(-1)^n\left(\prod_{i=0}^{k-1}(n^2-i^2)\right)q^{n^2}=\frac{(-1)^k}{(2k)!}\left(\prod_{i=0}^{k-1}(D-i^2)\right)\theta_2(q^2)\\ f_{2k+1}(q)&=\frac{(-1)^k}{(2k+1)!}\sum_{n\in\mathbb{Z}}(-1)^n(n+1/2)\left(\prod_{i=0}^{k-1}((n+1/2)^2-(i+1/2)^2)\right)q^{(n+1/2)^2}\\ &=\frac{(-1)^k}{(2k+1)!}\left(\prod_{i=0}^{k-1}(D-(i+1/2)^2)\right)\eta(q^2)^3. \end{aligned} \tag{5.4}$$

Here as before we denote $D=q\frac{d}{dq}$. In particular we have

$$f_0(q)=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2},\quad f_1(q)=\sum_{n\geq 0}(-1)^n(2n+1)q^{(n+1/2)^2},\quad f_2(q)=\sum_{n>0}(-1)^{n-1}n^2q^{n^2}.$$

We write $N^{(S,L),\delta}_{[k_1,\ldots,k_nr]}:=N^{(\widehat{S},L-k_1E_1-\ldots-k_rE_r),\delta}(y)$, to stress that we view it as a count of curves on S with prescribed multiplicities at the A_1-singularities.

Conjecture 39. *Let S be a toric surface with only A_1 singularities $p_1, \ldots, p_r$. Fix $k_1, \ldots, k_r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\delta \geq 0$. Let L be a Weil divisor on S, such that $L - \sum_i k_i E_i$ is a Cartier divisor on $\widehat{S}$, which is δ-very ample on any irreducible curve in $\widehat{S}$ not contained in $E_1 \cup \ldots \cup E_r$.. Then*

(5.5)
$$N^{(S,L),\delta}_{[k_1,\ldots,k_r]}(y) = \operatorname*{Coeff}_{q^{L(L-K_S)/2}}\left[\frac{\widetilde{DG}_2(y,q)^{\chi(L)-\sum_i k_i^2-1-\delta}B_1(y,q)^{K_S^2}}{B_2(y,q)^{LK_S}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_{i=1}^r f_{2k_i}(q)\right].$$

Thus we claim that the correction factors for points of multiplicity k at A_1 singularities of S are given by the quasimodular forms $f_k(q)$.

Equivalently we can look at the same question on the blowup $\widehat{S}$. Write $\widehat{L} := L - k_1E_1 - \ldots - k_rE_r$ and

$$\overline{f}_k(q) = \frac{f_k(q)}{q^{k^2/4}}, \quad k \in \frac{1}{2}\mathbb{Z}_{\geq 0},$$

then (with the same assumptions) (5.5) is clearly equivalent to

(5.6)
$$N^{(\widehat{S},\widehat{L}),\delta}(y) = \operatorname*{Coeff}_{q^{\widehat{L}(\widehat{L}-K_{\widehat{S}})/2}}\left[\frac{\widetilde{DG}_2(y,q)^{\chi(\widehat{L})-1-\delta}B_1(y,q)^{K_{\widehat{S}}^2}}{B_2(y,q)^{\widehat{L}K_{\widehat{S}}}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\prod_{i=1}^r \overline{f}_{2k_i}(q)\right].$$

In other words, the correction factors for $\widehat{L}$ not being sufficiently ample on $\widehat{S}$ are the $\overline{f}_l(q)$.

Remark 40. Under the assumptions of the conjecture, if the k_i are sufficiently large with respect to δ, then $\widehat{L}$ will be δ-very ample on $\widehat{S}$. This means by Conjecture 28 that for large l the correction factor $\overline{f}_l(q)$ should be 1 modulo some high power of q. In fact we find the following.

For $l \in \mathbb{Z}_{>0}$ we can rewrite

$$\overline{f}_l(q) = \sum_{m\geq 0}(-1)^m\frac{2m+l}{m+l}\binom{m+l}{l}q^{m(m+l)}.$$

In particular $\overline{f}_l(q) \equiv 1 \mod q^{l+1}$.

Proof. First we deal with the case l even. Note that

$$\prod_{i=0}^{k-1}(n^2-i^2) = n\prod_{i=-k-1}^{k-1}(n-i).$$

Thus we get for $k>0$

$$\overline{f}_{2k}(q) = \frac{(-1)^k}{(2k)!}\sum_{n\in\mathbb{Z}}(-1)^n\prod_{i=0}^{k-1}(n^2-i^2)q^{n^2-k^2} = \sum_{n\geq k}(-1)^{n-k}\frac{2n}{2k}\binom{n+k-1}{2k-1}q^{n^2-k^2},$$

where we also have used that $\binom{n+k-1}{2k-1}=0$ for $n<k$. Finally put $m=n-k$, so that $\frac{2n}{2k}\binom{n+k-1}{2k-1}=\frac{2m+2k}{m+2k}\binom{m+2k}{2k}$ and $n^2-k^2=m(m+2k)$.

The case l odd is similar. Note that

$$\prod_{i=0}^{k-1}((n+1/2)^2-(i+1/2)^2)=\prod_{i=-k+1}^{k}(n-i).$$

Thus we get

$$\begin{aligned}\overline{f}_{2k+1}(q)&=\frac{(-1)^k}{(2k+1)!}\sum_{n\geq 0}(-1)^n(2n+1)\left(\prod_{i=0}^{k-1}((n+1/2)^2-(i+1/2)^2)\right)q^{(n+1/2)^2-(k+1/2)^2}\\&=\sum_{n\in\mathbb{Z}}(-1)^{n-k}\frac{2n+1}{2k+2}\binom{n+k}{2k}q^{(n+1/2)^2-(k+1/2)^2},\end{aligned}$$

and put again $m:=n-k$. □

Remark 41. It is again remarkable that the correction factors $f_k(q)$ are independent of the variable y. In particular this means again that the correction factor is the same for the Severi degrees and for the tropical Welschinger number.

We specialise the conjecture to case that S is the weighted projective space $\mathbb{P}(1,1,2)$ with the resolution Σ_2 with more precise bounds for the validity. Note that

$$\chi(\Sigma_2,dH-kE)=(d+1)^2-k^2,\quad (dH-kE)K_{\Sigma_2}=(dH-kE)(-2H)=-4d,\quad K_{\Sigma_2}^2=8.$$

Conjecture 42. *Let $d,k\in\frac{1}{2}\mathbb{Z}$ with $d-k\in\mathbb{Z}$. Then for $\delta\leq 2(d-k)+1$, we have*

(5.7)
$$N^{(\Sigma_2,dH-kE),\delta}(y)=\operatorname*{Coeff}_{q^{d^2+2d-k^2}}\left[\frac{\widetilde{DG}_2(y,q)^{d^2+2d-k^2-\delta}B_1(y,q)^8}{B_2(y,q)^{4d}}\left(\frac{D\widetilde{DG}_2(y,q)}{\widetilde{\Delta}(y,q)}\right)^{1/2}\overline{f}_{2k}(q)\right].$$

Proposition 43. *(1) Conjecture 42 is true for all d, all $k\leq 5$ and $\delta\leq 4$.*
(2) The equation (5.7) holds for all $d,k\geq 0$ with $\delta\leq d-k$ and $\delta\leq 4$.

Proof. We use the Caporaso-Harris recursion to compute $N^{(\Sigma_2,dH+cF),\delta}(y)=N^{(\Sigma_2,(d+c/2)H-c/2E),\delta}(y)$ for $\delta\leq 8$, $d\leq 6$ and $c\leq 5$. We find in this realm that $N^{(\Sigma_2,(nH-kE),\delta}(y)$ is equal to the right hand side of Conjecture 42 for $\delta\leq 2(n-k)+1$. By Theorem 24 $Q^{(\Sigma_2,dH+cF),\delta}(y)$ is for fixed $c\geq 0$ and for $d\geq\delta$ a polynomial of degree 2 in d. Thus the above computations determine this polynomial for $\delta\leq 4$, and $c\leq 5$. On the other hand in dependence of c and d we have that $Q^{(\Sigma_2,dH+cF),\delta}(y)$ is for $c,d\geq\delta$ a polynomial in c and d of degree 2 in d and 1 in c. By the above we know this polynomial as a polynomial in d for $c=4$ and $c=5$. Thus it is determined and the claim follows. □

6. Counting curves with prescribed multiple points

Let S be a smooth projective surface, let $p_1,\ldots,p_r$ be general points on S, and let $\widehat{S}$ be the blowup of S in the p_i with exceptional divisors E_i. Let $n_1,\ldots,n_r\in\mathbb{Z}_{\geq 1}$. Let L be a sufficiently ample line bundle on S, and denote by the same letter its pullback to $\widehat{S}$. Note that $N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(1)$ counts the complex curves on S in $|L|$ with points of multiplicity n_i in p_i which have in addition δ nodes and pass through $\dim(|L-\sum_i n_iE_i)|)-\delta$ general points of S. If L is sufficiently ample, then the multiple points at the p_i impose $\sum_i \binom{n_i+1}{2}$ independent conditions on curves in $|L|$. Furthermore we see that

$$\chi(L-\sum_i n_iE_i)=\chi(L)-\sum_i \binom{n_i+1}{2}.$$

Now assume that S is a smooth projective toric surface. Let the $p_i\in S$ be fixed points of the torus action, so that $\widehat{S}$ is again a toric surface and the exceptional divisors E_i are torus-invariant divisors. Then by the above we can view $N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(y)$ as a refined count of curves in $|L|$ on S with points of multiplicity n_i at p_i for all i and in addition δ nodes which pass through

$$\dim(|L|)-\delta-\sum_i \binom{n_i+1}{2}$$

general points on S.

Notation 44. We denote $N^{(S,L),\delta}_{n_1,\ldots,n_r}(y):=N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(y)$.

For an Eisenstein series $G_{2k}(q)$, we denote

$$\overline{G}_k(q):=G_k(q)-G_k(q^2)=\sum_{n>0}\sum_{\substack{d|n\\ \frac{n}{d}\text{ odd}}} d^{2k-1}q^n.$$

We write again $D:=q\frac{\partial}{\partial q}$. Note that $D^lG_{2k}(q)$ and $D^l\overline{G}_{2k}(q)$ are quasimodular forms of weight $2k+2l$.

Conjecture 45. *For each $i\geq 1$ there exists a universal power series $H_i\in\mathbb{Q}[y^{\pm1}][[q]]$, such that, whenever L be sufficiently ample with respect to δ, r and $n_1,\ldots,n_r$, we have*

(6.1)

$$N^{(S,L),\delta}_{n_1,\ldots,n_r}(y)=$$

$$\operatorname*{Coeff}_{q^{(L^2-LK_S)/2}}\left[\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta-\sum_i\binom{n_i+1}{2}}\frac{B_1(y,q)^{K_S^2}B_2(y,q)^{LK_S}D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}}\prod_{i=1}^r H_{n_i}(y,q)\right].$$

Furthermore we conjecture for all $m>0$ the following:

1. *$H_m(y,q)$ can be expressed in terms of Jacobi theta functions and quasimodular forms.*
2. *$H_m(1,q)$ is a (usually non-homogeneous) polynomial in the $D^lG_{2k}(q)$ of weight $\leq 4k$.*
3. *$H_m(-1,q)$ is a (usually non-homogeneous) polynomial in the $D^lG_{2k}(q),D^l\overline{G}_{2k}(q)$ of weight $\leq 2k$.*

For small m we explicitly conjecture the following formulas:

(1) For $m \leq 2$ we conjecture

$$H_1(y,q) = \widetilde{DG}_2(y,q), \quad H_2(y,q) = \frac{F_1(y,q)}{(y^{1/2}-y^{-1/2})^4} + \frac{F_2(y,q)}{(y^{1/2}-y^{-1/2})^2(y-y^{-1})},$$

with

$$F_1(y,q) = \sum_{n>0}\sum_{d|n} \frac{1}{2}\left(-\frac{n^3}{d^3}+\frac{n^2}{d}-\frac{n}{d}\right)(y^{d/2}-y^{-d/2})^2 q^n$$

$$F_2(y,q) = \sum_{n>0}\sum_{d|n} \left(\frac{n^2}{d^2}-\frac{n}{2}\right)\frac{y^d-y^{-d}}{y-y^{-1}} q^n.$$

(2) For the specialisation at $y=1$ we conjecture the following (dropping the q from the notation).

$$\begin{aligned}
H_1(1) &= DG_2,\\
H_2(1) &= -\frac{1}{24}DG_2 + \frac{1}{6}D^2G_2 - \frac{1}{8}DG_4 - \frac{1}{24}D^3G_2 + \frac{1}{24}D^2G_4\\
H_3(1) &= \frac{DG_2}{90} - \frac{D^2G_2}{18} + \frac{DG_4}{24} - \frac{13D^3G_2}{288} - \frac{73D^2G_4}{1440} + \frac{DG_6}{120} - \frac{D^4G_2}{144} + \frac{13D^3G_4}{1440}\\
&\quad - \frac{D^2G_6}{480} + \frac{D^5G_2}{2880} - \frac{D^4G_4}{2016} + \frac{D^3G_6}{6912} + \frac{\Delta}{241920}\\
H_4(1) &= -\frac{9DG_2}{1120} + \frac{7D^2G_2}{160} - \frac{21DG_4}{640} - \frac{1063D^3G_2}{23040} + \frac{1207D^2G_4}{23040} - \frac{3DG_6}{320} + \frac{79D^4G_2}{5760}\\
&\quad - \frac{43D^3G_4}{2304} + \frac{149D^2G_6}{26880} - \frac{DG_8}{2688} - \frac{91D^5G_2}{69120} + \frac{95D^4G_4}{48384} - \frac{461D^3G_6}{645120} + \frac{101D^2G_8}{1451520}\\
&\quad - \frac{11\Delta}{5806080} + \frac{D^6G_2}{17280} - \frac{89D^5G_4}{967680} + \frac{D^4G_6}{25920} - \frac{D^3G_8}{207360} + \frac{D\Delta}{2903040} - \frac{D^7G_2}{967680}\\
&\quad + \frac{D^6G_4}{580608} - \frac{D^5G_6}{1244160} + \frac{D^4G_4}{8211456} - \frac{D^2\Delta}{84913920} + \frac{\Delta G_4}{864864}
\end{aligned}$$

(3) At $y=-1$ we conjecture

$$\begin{aligned}
H_1(-1) &= \overline{G}_2(q),\\
H_2(-1) &= \frac{1}{8}\big(\overline{G}_2 - D\overline{G}_2 + \overline{G}_4 - DG_2\big),\\
H_3(-1) &= \frac{1}{24}\overline{G}_2 - \frac{1}{24}DG_2 + \frac{7}{96}\overline{G}_4 - \frac{7}{96}D\overline{G}_2 + \frac{1}{2}\overline{G}_2^3 - \frac{1}{192}D\overline{G}_4 - \frac{5}{64}G_4\overline{G}_2 + \frac{1}{96}D^2G_2\\
&\quad - \frac{5}{1024}DG_4,\\
H_4(-1) &= \frac{3\overline{G}_2}{128} - \frac{5DG_2}{192} - \frac{67D\overline{G}_2}{1536} + \frac{67\overline{G}_4}{1536} + \frac{35D^2G_2}{2304} - \frac{247DG_4}{24576} + \frac{55\overline{G}_2^3}{144} - \frac{55G_4\overline{G}_2}{1536}\\
&\quad - \frac{11D\overline{G}_4}{4608} + \frac{D^3G_2}{192} + \frac{25D^2G_4}{6144} - \frac{7DG_6}{8192} + \frac{11\overline{G}_2^4}{8} - \frac{13\overline{G}_2D^2G_2}{192} + \frac{35\overline{G}_2DG_4}{512}\\
&\quad - \frac{21G_6\overline{G}_2}{1024} + \frac{D^2\overline{G}_4}{512}.
\end{aligned}$$

Remark 46. Part (1) of Conjecture 45 is not formulated in a very precise way. We want to illustrate the statement for $H_1(y,q)$ and $H_2(y,q)$, which we have conjecturally determined. Writing $\widetilde{DG}_2(y,q)=\frac{F_0(y,q)}{y-2+y^{-1}}$ we have

$$F_0(y,q)=-\frac{D\theta(y)}{\theta(y)}-3G_2,$$
$$F_1(y,q)=\frac{1}{2}\frac{(D\theta(y))^2}{\theta(y)^2}+3\frac{D\theta(y)}{\theta(y)}G_2+\frac{1}{2}\frac{D\theta(y)}{\theta(y)}+\frac{15}{8}G_4-\frac{9}{4}DG_2+\frac{3}{2}G_2,$$
$$F_2(y,q)=-\frac{1}{2}\frac{D\theta(y)\theta'(y)}{\theta(y)^2}-\frac{1}{6}\frac{D\theta'(y)}{\theta(y)}-2G_2\frac{\theta'(y)}{\theta(y)}.$$

Proof. A similar computation has been done in [GS2, Rem 1.4]. By definition we have

$$F_0(y,q)=\sum_{m>0}\sum_{d>0}m(y^d-2+y^{-d})q^{md}=\sum_{md>0}my^dq^{md}-2G_2(q)+\frac{1}{12}.$$

In [Z, page 456, compare (iii) and (vii)] it is proved that

$$\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)}=\frac{wy-1}{(w-1)(y-1)}-\sum_{nd>0}\operatorname{sgn}(d)w^ny^dq^{nd}. \tag{6.2}$$

Write $w=e^x$ and take the coefficient of x on both sides of (6.2). By the identity [Z, eq. (7)] we have

$$\frac{x\theta'(0)}{\theta(w)}=\exp\left(2\sum_{k\geq 2}G_k(q)\frac{z_1^k}{k!}\right).$$

This gives

$$\operatorname*{Coeff}_{x}\left[\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)}\right]=\operatorname*{Coeff}_{x^2}\left[\frac{\theta(wy)}{\theta(y)}\right]+G_2(\tau)=\frac{1}{2}\frac{\theta''(y)}{\theta(y)}+G_2(\tau)=\frac{D\theta(y)}{\theta(y)}+G_2(\tau),$$

where the last step is by the heat equation $\frac{1}{2}\theta''(y)=D\theta(y)$. On the other hand we compute

$$\operatorname*{Coeff}_{z_1}\left[\frac{wy-1}{(w-1)(y-1)}-\sum_{nd>0}\operatorname{sgn}(d)w^ny^dq^{nd}\right]=\frac{1}{12}-\sum_{nd>0}ny^dq^{nd}.$$

This proves the formula for F_0.

We have

$$F_2(y,q)=\sum_{md>0}\operatorname{sgn}(d)(m^2-md/2)y^d)q^{md}.$$

In [GS2, Rem. 1.4] it is shown (the statement there contains a misprint) that

$$\sum_{md>0}\operatorname{sgn}(d)m^2y^dq^{md}=-\frac{1}{\theta(y)}\left(\frac{2}{3}D\theta'(y)+2G_2(q)\theta'(y)\right).$$

We see by (6.2) that

$$\sum_{md>0} \operatorname{sgn}(d)(-md/2)y^d)q^{md} = \frac{1}{2}D\left(\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)}\Big|_{w=1}\right) = \frac{1}{2}D\left(\frac{\theta'(y)}{\theta(y)}\right).$$

This shows the formula for F_2.

A similar but slightly more tedious computation shows the formula for F_1. □

The conjectural formulas of Conjecture 45 were found by doing computations for $\mathbb{P}^2$ and its blowup Σ_1 with exceptional divisor E. We use the Caporaso Harris recursion formula to compute $N^{(\Sigma_1,dH+mF),\delta}(y) = N^{(\Sigma_1,(d+m)H-mcE,\delta}$ for $d \le 11$, $m \le 4$ and $\delta \le 22$, in this realm the following conjecture is true.

Conjecture 47. *There are power series $H_m(y,q) \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that the following holds. For $d > 0$, and $0 \le m \le 4$ and $\delta \le 2d+1+m(m+1)/2$ we have*

$$N_m^{(\mathbb{P}^2,dH),\delta}(y) =$$

$$\operatorname*{Coeff}_{q^{(d(d+3)/2}}\left[\widetilde{DG}_2(y,q)^{d(d+3)/2-m(m+1)/2-\delta}\frac{B_1(y,q)^9(D\widetilde{DG}_2(y,q))^{1/2}}{B_2(y,q)^{-3d}\widetilde{\Delta}(y,q)^{1/2}}H_m(y,q)\right].$$

Furthermore $H_1(y,q)$, $H_2(y,q)$ coincide with the functions with the same name from Conjecture 45, and $H_i(1,q)$, $H_i(-1,q)$ coincide for $i = 1,2,3,4$ with the $H_i(1)$, $H_i(-1)$ from Conjecture 45.

Proposition 48. *Conjecture 47 is true from $m \le 4$ and $\delta \le 9$.*

Proof. The argument is the same as in several proofs before. By Theorem 24 we get that $Q^{(\Sigma_1,dH+mF),\delta}$ is for $\delta \le d$ a polynomial of degree 2 in d, which we know for $9 \le d \le 11$. The result follows. □

Let S be a toric surface and $\widehat{S}$ be the blowup of S in torus fixed point. Given δ, if m is sufficiently large and L is sufficiently ample on S, then $L - mE$ will be sufficiently ample on $\widehat{S}$, so that Conjecture 28 will apply to the pair $(\widehat{S}, L - mE)$:

$$N_m^{(S,L),\delta)}(y) = N^{(\widehat{S},L-mE),\delta}(y) =$$

$$\operatorname*{Coeff}_{q^{(L^2-LK_S)/2-\binom{m+1}{2}}}\left[\widetilde{DG}_2(y,q)^{\chi(L)-1-\delta-\binom{m+1}{2}}\frac{B_1(y,q)^{K_S^2-1}B_2(y,q)^{LK_S+m}D\widetilde{DG}_2(y,q)}{(\widetilde{\Delta}(y,q)\cdot D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_S)/2}}\right].$$

Combined with Conjecture 45 this leads to the following conjecture.

Conjecture 49. *We have*

$$\frac{H_m(y,q)}{q^{\binom{m+1}{2}}} \equiv \frac{B_2(y,q)^m}{B_1(y,q)} \mod q^{m+1}.$$

Thus, if eventually one would find a way to explicitly determine the functions $H_m(y,q)$ for all m, this could give the unknown power series $B_1(y,q)$, $B_2(y,q)$ and thus complete the conjectural formulas of [Göt],[GS].

It is natural to assume that the specialisation of Conjecture 45 and also of the previous conjectures Conjecture 34, Conjecture 39 to $y=1$ hold for the usual Severi degrees $n^{(S,L),\delta}$ for projective algebraic surfaces, not just for toric surfaces. Thus we get in particular the following generalisation of the original conjecture of [Göt].

Let S be a projective algebraic surface with A_1-singularties $q_1,\ldots,q_s$. Let $p_1,\ldots p_r$ be distinct smooth points on S. Let $m_1,\ldots,m_r \in \mathbb{Z}_{>0}$, $n_1,\ldots,n_s \in \mathbb{Z}_{\geq 0}$. Let $\widehat{S}$ be the blowup of S in $q_1,\ldots,q_s,p_1,\ldots p_r$ and denote E_i, F_j the exceptional divisors over q_i, p_j respectively. Let L be a $\mathbb{Q}$-Cartier Weil divisor on S, such that $\widehat{L} := L - \sum_{i=1}^s m_i E_i - \sum_{i=1}^r n_i F_i$ is a Cartier divisor on $\widehat{S}$, which is δ-very ample on all irreducible curves in $\widehat{S}$ not contained in $E_1\cup\ldots\cup E_s\cup F_1\cup\ldots\cup F_r$. Denote $n^{(S,L),\delta}_{(m_1,\ldots,m_r),(n_1,\ldots,n_s)} := n^{(\widehat{S},\widehat{L}),\delta}$, which we could informally interpret as the number of curves in $|L|$ which have multiplicity m_i in p_i and n_j in q_j for all i,j and pass in addition through

$$\dim|L| - \sum_{i=1}^r \binom{m_i+1}{2} - \sum_{j=1}^s \frac{n_j^2}{4}$$

general points on S, and have δ nodes as other singularities.

Conjecture 50.

(6.3)

$$n^{(S,L),\delta}_{(m_1,\ldots,m_r),(n_1,\ldots,n_s)} = \operatorname*{Coeff}_{q^{(L^2-LK_S)/2}} \left[DG_2(q)^{\chi(L)-\sum_i \binom{m_i+1}{2} - \sum_j \frac{n_j^2}{4} - 1} \frac{B_1(q)^{K_S^2} B_2(q)^{LK_S} D^2G_2(q)}{(\Delta(q)\cdot D^2G_2(q))^{\chi(\mathcal{O}_S)/2}} \left(\prod_{i=1}^r H_{n_i}(1,q)\right)\left(\prod_{i=1}^s f_{m_i}(q)\right)\right].$$

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